DECORATED HYPERTREES

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ABSTRACT. C. Jensen, J. McCammond and J. Meier have used weighted hypertrees to compute the Euler characteristic of a subgroup of the automorphism group of a free product. Weighted hypertrees also appear in the study of the homology of the hypertree poset. We link them to decorated hypertrees after a general study on decorated hypertrees, which we enumerate using box trees.

CONTENTS

INTRODUCTION

The notion of hypertree has been introduced by C. Berge in his book [Ber89] during the 1980's. Recently, the hypertree poset has been used by C. Jensen, J. McCammond and J. Meier in their articles [MM04], [JMM07a] and [JMM06] for the study of a subgroup of the automorphism group of the free groups. The characteristic polynomial of the poset has been computed by F. Chapoton in the article [Cha07] and we have studied, in the article [Oge12], the character of the action of the symmetric group on the Whitney homology of the hypertree poset.

In the article [JMM07b], C. Jensen, J. McCammond and J. Meier have used a kind of decorated hypertrees to compute the Euler characteristic of a subgroup of the automorphism group of a free product. In his article [Cha07], F. Chapoton has introduced cyclic hypertrees, which are a type of hypertrees whose vertices have their neighbourhood decorated by the species of cycles. Moreover, in the article [Oge12], the character of the action of the symmetric group on the Whitney homology of the hypertree poset was related to a series which seems to be the cycle index series of a species of decorated hypertrees.

In this context, the study of decorated hypertrees seems natural. In the first part of this article, we recall the definition of hypergraphs and hypertrees and we define the species of edge-decorated hypertrees and its pointed variations before giving relations between them. The second part is the enumeration of edge-decorated hypertrees using what we call box trees. After this general study, we examine the case of $\widehat{\mathrm{PreLie}}$-decorated hypertrees and $\widehat{\mathrm{Lie}}$-decorated hypertrees by computing

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their cycle index series. The case of $\widehat{\text{PreLie}}$-decorated hypertrees appears in the article [JMM07b] and is related to 2-coloured trees. Both cases are linked with an operad.

In the last part, we generalize edge-decorated hypertrees by also decorating neighbourhoods of vertices. Cyclic hypertrees introduced in [Cha07] are a particular case of these hypertrees. Moreover, in this type of decorated hypertrees also fit the decorated hypertrees which are related to the Whitney homology of the hypertree poset.

We use the notation $\#E$ for the cardinality of a finite set E. We will use the language of species, defined in 1.15 and in the book of F. Bergeron, G. Labelle and P. Leroux [BLL98].

We particularly thank V. Dotsenko for his useful proof of the Koszulness of the operad Λ.

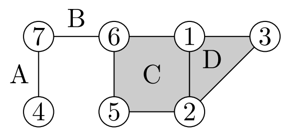
1. Description and relations of the edge-decorated hypertrees

In this section, we introduce a type of weighted hypertrees, named *edge-decorated hypertrees* and give functional equations satisfied by them.

1.1. From hypergraphs to rooted and pointed hypertrees. We first recall the definition of hypergraphs and hypertrees.

Definition 1.1. An *hypergraph (on a set V)* is an ordered pair (V, E) where V is a finite set and E is a collection of parts of V of cardinality at least two. The elements of V are called *vertices* and those of E are called *edges*.

Example 1.2. An example of hypergraph on $\{1, 2, 3, 4, 5, 6, 7\}$.



Definition 1.3. Let $H = (V, E)$ be an hypergraph, v and w two vertices of H. A *walk from v to w in H* is an alternating sequence of vertices and edges $(v = v_1, e_1, v_2, \ldots, e_n, v_{n+1} = w)$ where for all i in $\{1, \ldots, n+1\}$, $v_i \in V$, $e_i \in E$ and for all i in $\{1, \ldots, n\}$, $\{v_i, v_{i+1}\} \subseteq e_i$.

Example 1.4. In the example 1.2, there are several walks from 4 to 2:
$(4, A, 7, B, 6, C, 2)$ and $(4, A, 7, B, 6, C, 1, D, 2)$.

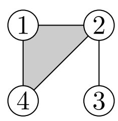
In this article, we are interested in a special type of hypergraphs: hypertrees.

Definition 1.5. An *hypertree* is a non empty hypergraph H such that, given any vertices v and w in H,

- there exists a walk from v to w in H with distinct edges e_i, i.e. H is *connected*,
- and this walk is unique, i.e. H has *no cycles*.

The pair $H = (V, E)$ is called *hypertree on V*. If V is the set $\{1, \ldots, n\}$, then H is called an *hypertree on n vertices*.

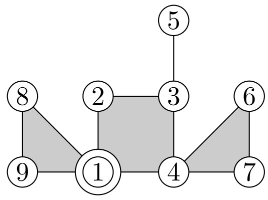
Example 1.6. An example of hypertree on $\{1, 2, 3, 4\}$.



We now recall rooted and pointed variations of hypertrees.

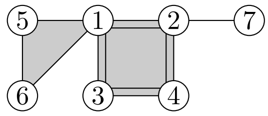
Definition 1.7. A *rooted hypertree* is an hypertree H together with a vertex s of H. The hypertree H is said to be *rooted at* s and s is called the root of H.

Example 1.8. An hypertree on nine vertices, rooted at 1.



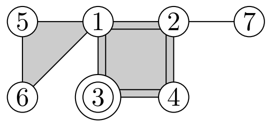
Definition 1.9. An *edge-pointed hypertree* is an hypertree H together with an edge e of H. The hypertree H is said to be *pointed at* e.

Example 1.10. An hypertree on seven vertices, pointed at the edge $\{1, 2, 3, 4\}$.



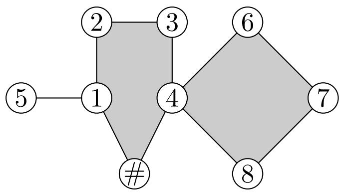
Definition 1.11. An *edge-pointed rooted hypertree* is an hypertree H on at least two vertices, together with an edge a of H and a vertex v of a. The hypertree H is said to be *pointed at* a *and rooted at* s.

Example 1.12. An hypertree on seven vertices, pointed at edge $\{1, 2, 3, 4\}$ and rooted at 3.



Definition 1.13. A *hollow hypertree* is an hypertree on at least two vertices, in which one vertex, belonging to one and only one edge, has been cancelled and replaced by a gap #.

Example 1.14. Hollow hypertree on eight vertices. The hollow edge is the edge $\{1, 2, 3, 4\}$.



1.2. **Definitions of decorated hypertrees.** From an hypertree, we can define what we call an *edge-decorated hypertree*. This definition uses the following notion of *species*:

Definition 1.15. A *species* F is a functor from the category of finite sets and bijections to the category of finite sets. To a finite set I, the species F associates a finite set $\mathrm{F}(I)$ independent from the nature of I.

Example 1.16.
- The map which associates to a finite set I the set of total orders on I is a species, called the linear order species and denoted by Assoc.
- The map which associates to a finite set I the set $\{I\}$ is a species, called the set species and denoted by $\mathcal{E}$.

- The map which associates to a finite set I the set of cyclic orders on I is a species, called the cycle species and denoted by Cycle.
- The map which associates to a finite set I the set of hypertrees on I is a species, called the hypertree species and denoted by $\mathcal{H}$.
- The map which associates to a finite set I the set I is a species, called the the pointed set species and denoted by Perm.
- The map defined for every finite set I by:

$$I \mapsto \begin{cases} \{I\} & \text{if } \#I \geq 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

  is a species denoted by Comm.
- The map defined for every finite set I by:

$$I \mapsto \begin{cases} \{I\} & \text{if } \#I = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

  is a species, called the singleton species and denoted by X.

The map which associates to a finite set I the set of rooted (resp. edge-pointed, edge-pointed rooted, hollow) hypertrees on I is a species, called the rooted (resp. edge-pointed, edge-pointed rooted, hollow) hypertree species and denoted by $\mathcal{H}^p$ (resp. $\mathcal{H}^a$, $\mathcal{H}^{pa}$, $\mathcal{H}^c$).

Definition 1.17 ([BLL98])**.** Let F and G be two species. An *isomorphism* of F to G is a family of bijections $\alpha_U : F(U) \mapsto G(U)$, where U is a finite set, such that for any finite set V, any bijection $\sigma : U \mapsto V$ and any F-structure $s \in F(u)$, the following equality holds:

$$\sigma \cdot \alpha_U(s) = \alpha_V(\sigma \cdot s)$$

where the $\cdot$ stands for the action of the symmetric group on the structure.

The two species F and G are then said to be isomorphic.

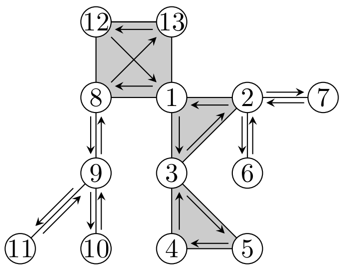
Having defined species, we can now give the definition of an *edge-decorated hypertree*.

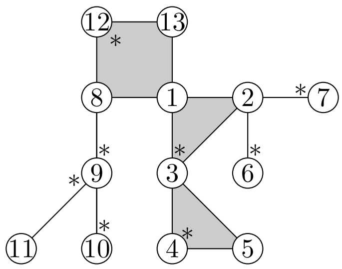
Definition 1.18. Given a species $\mathcal{S}$, an *edge-decorated (edge-pointed) hypertree* is obtained from an (edge-pointed) hypertree H by choosing for every edge e of H an element of $\mathcal{S}(V_e)$, where V_e is the set of vertices in the edge e.

Given a species $\mathcal{S}$, the map which associates to a finite set I the set of edge-decorated hypertrees on I is a species, called the $\mathcal{S}$-edge-decorated hypertree species and denoted by $\mathcal{H}_{\mathcal{S}}$. The map which associates to a finite set I the set of edge-decorated edge-pointed hypertrees on I is a species, called the $\mathcal{S}$-edge-decorated edge-pointed hypertree species and denoted by $\mathcal{H}^a_{\mathcal{S}}$. When the species used is obvious, we will omit to write it.

Example 1.19. We consider two different decorations of the same hypertree H.

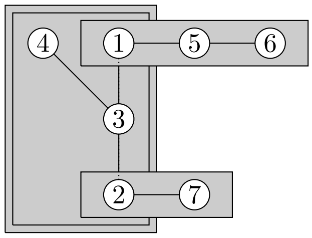
An edge-decorated hypertree obtained from H by decorating its edges with the cycle species Cycle is drawn in the left part below.





An edge-decorated hypertree obtained from H by decorating its edges with the pointed set species $Perm$ is drawn in the right part above. The vertex of each edge with a star $*$ next to it is the pointed vertex of the edge: for instance, the pointed vertex of $\{9, 11\}$ is 9, the pointed vertex of $\{9, 10\}$ is 10, the pointed vertex of $\{8, 9\}$ is 9 and the pointed vertex of $\{12, 13, 1, 8\}$ is 12.

We now decorate the edge-pointed hypertree of the example 1.10 with the species of trees, which associates to a finite set I the set of trees with vertex set I. The edges are represented by rectangles on the scheme, with the pointed edges represented by the double rectangle.



We now give definitions for rooted or hollow versions of edge-decorated hypertrees.

To define *edge-decorated rooted hypertrees*, we will need the following operation on species:

Definition 1.20. Let F be a species. The differential of F is the species defined as follow:

$$F'(I) = F(I \sqcup \{\bullet\}).$$

Example 1.21. Considering the examples of species given in the example 1.16, their differentials are the following:

- The differential of the cycle species Cycle is the linear order species Assoc.
- The differential of the set species $\mathcal{E}$ is $\mathcal{E}$.
- The differential of the pointed set species Perm is the species Perm$'$ such that, for all finite set I, the set Perm$'(I)$ is equal to the set $\mathcal{E}(I) \cup \text{Perm}(I)$.

We can now define the decoration for edges of rooted variations of hypertrees. There is a definition of edge-decorated rooted, edge-decorated edge-pointed rooted or edge-decorated hollow hypertrees analogous to the one of edge-decorated hypertrees and edge-decorated edge-pointed hypertrees:

Definition 1.22. Given a species $\mathcal{S}$, an *edge-decorated (edge-pointed) rooted hypertree* (resp. *edge-decorated hollow hypertree*) is obtained from an (edge-pointed) rooted (resp. hollow) hypertree H by choosing for every edge e of H an element of $\mathcal{S}(V_e)$, where V_e is the set of vertices in the edge e.

In rooted or hollow hypertrees, there is one distinguished vertex in every edge. Therefore, using the definition of the differential of a species, we obtain the following equivalent definition:

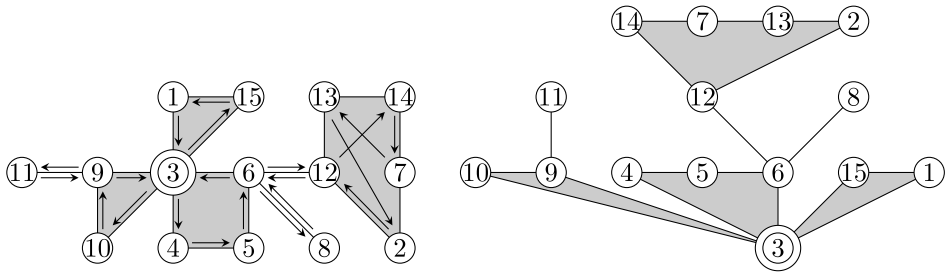
Definition 1.23. Let us consider a rooted (resp. edge-pointed rooted, resp. hollow) hypertree H. Given an edge e of H, there is one vertex of e which is the nearest from the root (resp. the gap) of H in e: let us call it the *petiole* of e. Then, an *edge-decorated rooted hypertree* (resp. *edge-decorated rooted edge-pointed hypertree*, resp. *edge-decorated hollow hypertree*) is obtained from the hypertree H by choosing for every edge e of H an element in the set $\mathcal{S}'(V_e^l)$, where the set V_e^l of vertices of e different from the petiole.

The map which associates to a finite set I the set of edge-decorated rooted hypertrees on I is a species, called the $\mathcal{S}$-edge-decorated rooted hypertree species and denoted by $\mathcal{H}^p_{\mathcal{S}}$.

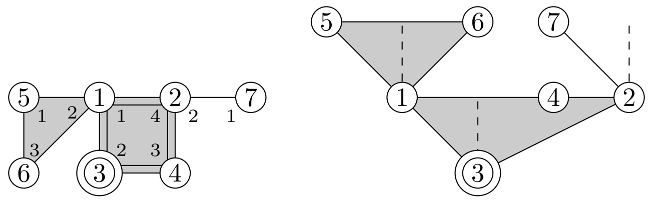
The map which associates to a finite set I the set of edge-decorated edge-pointed rooted hypertrees on I is a species, called the $\mathcal{S}$-edge-decorated edge-pointed rooted hypertree species and denoted by $\mathcal{H}^{pa}_{\mathcal{S}}$.

The map which associates to a finite set I the set of edge-decorated hollow hypertrees on I is a species, called the $\mathcal{S}$-edge-decorated hollow hypertree species and denoted by $\mathcal{H}^c_{\mathcal{S}}$.

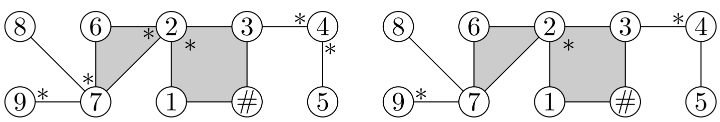
Example 1.24. We give an example of an edge-decorated rooted hypertree. The left-side picture below is the one obtained when edges are decorated by the cycles species. The right-side picture below is the same rooted hypertree in which the set of elements of an edge different from the petiole forms a list. The two pictures are two representations of the same hypertree.



We now decorate the edges of an edge-pointed rooted hypertree with the species of lists, represented on the left-side below. The small numbers inside the edges near every vertex is the number of the vertex in the list. The derivative of the species of lists is the species of the unions of two lists. Hence this decorated hypertree is equivalent to the following right-side hypertree where, for all edges, the set of vertices different from the petiole is separated into two lists. We draw a dashed line for the separation in an edge between the two lists. In the edge $\{2, 7\}$, the second list is empty.



We now give an example of an edge-decorated hollow hypertree decorated by the species of non-empty pointed sets Perm. As previously, the left part of the example is obtained with the first definition of edge-decorated hollow hypertrees. We draw a star $*$ next to the pointed vertex. The right part is obtain with the second definition.



We now generalize this definitions to linear species.

1.3. **Generalization to linear species and first relation with other trees.** The definition 1.23 motivates the introduction of rooted or hollow hypertrees with edges decorated by species which are not the differential of a species but of a linear species.

Definition 1.25. A *linear species* F is a functor from the category of finite sets and bijections to the category of vector spaces. To a finite set I, the species F associates a vector space $\mathrm{F}(I)$ independent from the nature of I.

We define the differential of linear species:

Definition 1.26. Let F be a linear species. The differential of F is the species defined as follow:

$$F'(I) = F(I \sqcup \{\bullet\}).$$

We can define the same operations as for species on linear species.

We can now generalize the decoration of edges of hypertrees:

Definition 1.27. Given a species $\mathcal{S}$, an *edge-decorated (edge-pointed) hypertree* is obtained from an (edge-pointed) hypertree H by choosing for every edge e of H an element of $\mathcal{S}(V_e)$, where V_e is the set of vertices in the edge e.

This decoration is multi-linear.

We define similarly edge-decorated (edge-pointed) rooted hypertrees and edge-decorated hollow hypertrees.

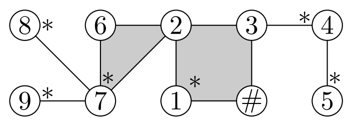
Let us consider a species $\mathcal{S}$. When there exists a species or a linear species $\mathcal{F}$ such that $\mathcal{F}' = \mathcal{S}$, we denote it by $\widehat{\mathcal{S}}$.

For every cyclic or anticyclic operad $\mathcal{C}$, there always exists a linear species $\widehat{\mathcal{C}}$. This proves the existence of the linear species $\widehat{\mathrm{PreLie}}$, $\widehat{\mathrm{Lie}}$, $\widehat{\mathrm{Perm}}$ and $\widehat{\mathrm{Assoc}}$. This case of operads is examined in the article of F. Chapoton [Cha05] and the article of E. Getzler and M. Kapranov [GK95].

The notion of edge-decorated hollow hypertrees can be related with different objects according to the decoration. For instance, if we consider the linear species $\widehat{\mathrm{Perm}}$ whose derivative is the species Perm, the associated edge-decorated hollow hypertrees will be related with what is called fat trees.

We first give an example of $\widehat{\mathrm{Perm}}$-edge-decorated hollow hypertree, where Perm is the species of non-empty pointed sets. The pointed vertex of each edge is marked by a star $*$ next to it. The petiole of the edge $\{2, 6, 7\}$ is 2. Remark that vertices 8, 9, 4 and 5 are necessarily pointed because they are alone with the petiole in their edge.

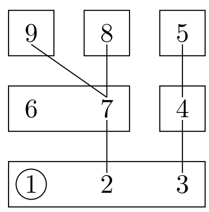
Example 1.28. An example of $\widehat{\mathrm{Perm}}$-edge-decorated hollow hypertree



Definition 1.29 ([Zas02]). A *fat tree on a set V* is a partition of V, whose parts are called *vertices*, together with edges linking elements of different vertices, in a tree-like manner.

A *rooted fat tree* is a fat tree with a distinguished element called the *root*.

Example 1.30. An example of a rooted fat tree. The root is circled.



Proposition 1.31. *The species of* $\widehat{\mathrm{Perm}}$*-edge-decorated hollow hypertrees is isomorphic to the species of rooted fat trees.*

Proof. First, remark that $\widehat{\mathrm{Perm}}$-edge-decorated hollow hypertrees are hollow hypertrees where in every edge, the set of vertices different from the petiole is pointed.

Let us consider a rooted fat tree FT on a finite set V. We call *petiole* of an edge e the closest vertex of e from the root and *end* the other vertex.

Let us consider a vertex v of the rooted fat tree. We form an edge of an hypergraph by considering the set of elements of V in the vertex v and the petiole of the edge linking this vertex to the root. Let us call E the set of such edges. The hypergraph $H = (E, V)$ thus obtained is an hypertree because every path in H was in FT. If we put a gap in the edge of E containing the elements of V that were in the root of FT, we obtain a hollow hypertree. Moreover, the end of every edge in FT gives a pointed element in the set of vertices of an edge of H different from the petiole. This structure is the same as the one obtained by decorating edges with $\widehat{\mathrm{Perm}}$.

Conversely, let us now take a $\widehat{\mathrm{Perm}}$-edge-decorated hollow hypertree H. We consider a structure whose vertices are the edges of H without their petiole and without the gap, with the pointed elements of every edge of H linking to their petiole by an edge in the new structure: this is a fat tree FT. By distinguishing the vertex of FT obtained from the hollow edge of H, we obtain a rooted fat tree.

Example 1.32. The two previous examples 1.28 and 1.30 are related by the bijection of the proof.

□

1.4. **Relations.**

1.4.1. *Dissymetry principle.*

The reader may consult the book [BLL98, Chapter 2.3] for a deeper explanation on the dissymmetry principle. In a general way, a *dissymmetry principle* is the use of a natural center to obtain the expression of a non pointed species in terms of pointed species. An example of this principle is the use of the center of a tree to express unrooted trees in terms of rooted trees.

We will consider the following weight on any hypertree (pointed or not, rooted or not, hollow or not):

Definition 1.33. The weight of an hypertree H with edge set E is given by:

$$W_t(H) = t^{\#E-1}.$$

The expression of the hypertree species in term of pointed and rooted hypertrees species is the following:

Proposition 1.34 ([Oge12])**.** *The species of hypertrees and the one of rooted hypertrees are related by:*

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}. \tag{1.1}$$

This bijection links hypertrees with k edges with hypertrees with k edges, and therefore it preserves the weight on hypertrees. Pointing a vertex or an edge of an hypertree and then decorating its edges is just the same as decorating its edges and then pointing a vertex or an edge. Therefore, the decoration of edges is compatible with the previous property 1.34 and we obtain:

Proposition 1.35 (Dissymmetry principle for edge-decorated hypertrees)**.** *Given a species $\mathcal{S}$, the following relation holds:*

$$\mathcal{H}_{\mathcal{S}} + \mathcal{H}_{\mathcal{S}}^{pa} = \mathcal{H}_{\mathcal{S}}^{p} + \mathcal{H}_{\mathcal{S}}^{a}. \tag{1.2}$$

1.4.2. *Functional equations.* To establish relations between species, we will need the following operations on species:

Definition 1.36. Let F and G be two species. We define the following operations on species:

- $(F+G)(I) = F(I) \sqcup G(I)$, (addition)
- $(F \times G)(I) = F(I) \times G(I)$, (product)
- $(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, (plethystic substitution) where $\mathcal{P}(I)$ runs on the set of partitions of I.

The previous species are related by the following proposition:

Proposition 1.37. *Given a species $\mathcal{S}$ such that $\#S(\emptyset) = 0$ and $\#S(\{1\}) = 1$, the species $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}^p_{\mathcal{S}}$, $\mathcal{H}^a_{\mathcal{S}}$, $\mathcal{H}^{pa}_{\mathcal{S}}$ and $\mathcal{H}^c_{\mathcal{S}}$ satisfy:*

$$\mathcal{H}^p_{\mathcal{S}} = X \times \mathcal{H}'_{\mathcal{S}}, \tag{1.3}$$

$$t\mathcal{H}^p_{\mathcal{S}} = X + X \times \mathrm{Comm} \circ (t \times \mathcal{H}^c_{\mathcal{S}}), \tag{1.4}$$

$$\mathcal{H}^c_{\mathcal{S}} = (S' - 1) \circ t\mathcal{H}^p_{\mathcal{S}}, \tag{1.5}$$

$$\mathcal{H}^a_{\mathcal{S}} = (S - X) \circ t\mathcal{H}^p_{\mathcal{S}}, \tag{1.6}$$

$$\mathcal{H}^{pa}_{\mathcal{S}} = \mathcal{H}^c_{\mathcal{S}} \times t\mathcal{H}^p_{\mathcal{S}} = X \times \left(X(1 + \mathrm{Comm}) \circ \mathcal{H}^c_{\mathcal{S}}\right). \tag{1.7}$$

Proof. If we multiply the series by t, the power of t corresponds to the number of edges in the associated hypertrees.

- The first relation is due to the relation between a species and a pointed species.
- The second one is obtained from a decomposition of a rooted hypertree. If there is only one vertex, we keep the label of it and it gives X, the number of edges is 0. Otherwise, we separate the label of the root: it gives X. It remains an hypertree with a gap contained in different edges. Separating these edges, we obtain a non-empty set of hollow hypertrees with edges decorated by $\mathcal{S}$. There is the same number of edges in the set of hollow hypertrees as in the rooted hypertree. This operation is a bijection of species because a vertex alone is a rooted edge-decorated hypertree and taking a non-empty forest of hollow edge-decorated hypertrees and linking them by their gap on which we put a label gives a rooted edge-decorated hypertree.
- The third relation is obtained by pointing the vertices in the hollow edge and breaking the edge: we obtain a non-empty forest of rooted edge-decorated hypertrees. As we break an edge, there is one edge less in the forest of rooted hypertrees than in the hollow hypertree: we can simplify the t in front of $\mathcal{H}^c_{\mathcal{S}}$. The set of roots is a $\mathcal{S}'$-structure and induces this structure on the set of trees: we obtain a $\mathcal{S}'$-structure in which all elements are rooted edge-decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is a bijection of species.
- The fourth relation is obtained by pointing the vertices in the pointed edge and breaking it: we obtain a non-empty forest of at least two rooted edge-decorated hypertrees. As we break an edge, there is one edge less in the forest of rooted hypertrees than in the edge-pointed hypertree: we

can simplify the t in front of $\mathcal{H}_{\mathcal{S}}^{a}$. The set of roots is a $\mathcal{S}$-structure and induces this structure on the set of tree: we obtain a $\mathcal{S}$-structure in which all elements are rooted edge-decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is a bijection of species.

- The last relation is obtained by separating the pointed edge from the other edges containing the root and putting a gap in the pointed edge instead of the root: the connected component of the pointed edge gives an edge-decorated hollow hypertree and the connected component containing the root gives a rooted edge-decorated hypertree. There is the same number of edges in the edge-pointed rooted hypertree as in the union of the hollow and the rooted hypertree.

□

Corollary 1.38. *Using the equations* (1.4) *and* (1.5) *of the previous proposition, we obtain:*

$$t\mathcal{H}_{\mathcal{S}}^{p} = X + X \times \mathrm{Comm} \circ (t \times S' \circ t\mathcal{H}_{\mathcal{S}}^{p}) \tag{1.8}$$

and

$$\mathcal{H}_{\mathcal{S}}^{c} = (S' - 1) \circ (X + X \times \mathrm{Comm} \circ (t \times \mathcal{H}_{\mathcal{S}}^{c}))\,. \tag{1.9}$$

2. Counting edge-decorated hypertrees using box trees

In this section, we count edge-decorated hypertrees and pointed variants of them using a new type of tree-like structure, called box trees.

2.1. **Box trees.** Let us consider a quadruple (L, V, R, E), where

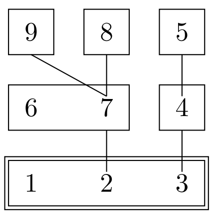
- L is a finite set called the set of *labels*,
- V is a partition of L called the set of *vertices*,
- R is an element of V called the root,
- E is a map from $V - \{R\}$ to L called the set of *edges*.

We will denote by $\tilde{E}$, the map from $V - \{R\}$ to V which associates to a vertex v the vertex v' containing the label $E(v)$. The pair $(V, \tilde{E})$ is then an oriented graph, with vertices labelled by subsets of L.

Definition 2.1. A quadruple (L, V, R, E) is a *box tree* if and only if $(V, \tilde{E})$ is a tree, rooted in R, with edges oriented toward the root.

A label l is called *father* of a vertex v if $E(v) = l$.

Example 2.2. An example of box tree. The root is the double rectangle.



The difference between box trees and fat trees is mainly in the edges, which are between labels for fat trees and a label and a vertex for box trees.

Proposition 2.3. *Let us consider L a finite set of cardinality n and V a partition of L into $k+1$ parts $p_0, p_1, \ldots, p_k$, where the cardinality of p_i is n_i. The number of box trees $N_{p_0;p_1,\ldots,p_k}$ on $k+1$ vertices with root labelled by p_0 and the other k vertices labelled by one of the k other p_i is given by:*

$$N_{p_0;p_1,\ldots,p_k} = n_0 \times n^{k-1}.$$

Proof. We prove this statement by induction on r.

For $r = 1$, there is only one vertex attached to a label of the root. As there is n_0 labels in the root, there is n_0 box trees on two vertices.

If this statement holds for all $k < r$, we compute the number $N_{p_0;p_1,\ldots,p_r}$ of box tree on $r+1$ vertices satisfying the hypothesis. It can be obtained by summing on the number of vertices attached to the root, called its *sons*. Let n be the total number of labels.

If the root has j sons, with respectively $n_{i_1}, \ldots, n_{i_j}$ labels, there are n_0^j ways of attaching them to the root. Moreover, cutting the root and gluing together the sons of the root in one vertex, we obtain a box tree on $r+1-j$ vertices and $n - n_0$ labels, with root having $n_{i_1} + \ldots + n_{i_j}$ labels. Using the induction hypothesis, we obtain:

$$N_{p_0;p_1,\ldots,p_r} = \sum_{j=1}^{r} n_0^j \times \sum_{i_1<\ldots<i_j} (n_{i_1} + \ldots + n_{i_j}) \times (n - n_0)^{r-j-1}.$$

In the second sum $\sum_{i_1<\ldots<i_j}(n_{i_1} + \ldots + n_{i_j})$, every n_i, with i different from 0, appears $\binom{r-1}{j-1}$ times, for $i \geq 1$. Indeed, in this case the vertex labelled by p_i is a son of the root so we choose the $j-1$ other sons among the $r-1$ other vertices. Therefore, we have:

$$N_{p_0;p_1,\ldots,p_r} = \sum_{j=1}^{r} \binom{r-1}{j-1} n_0^j \times (n - n_0)^{r-j} = n_0 \times (n)^{r-1}.$$

This gives the expected result. □

2.2. **Enumeration of decorated hypertrees.**

Theorem 2.4. *Given a species $\mathcal{S}$, every rooted hypertree with k edges and n vertices, whose edges are decorated by $\mathcal{S}$, can be decomposed as a triple $(r, \mathbb{S}, BT)$ where:*

- *r is the root of the hypertree,*
- *$\mathbb{S}$ is a set of k sets on $n-1$ vertices with an $\mathcal{S}'$-structure on each of them,*
- *and BT is a box tree on $k+1$ vertices with root labelled by r and the other vertices labelled by one of the k sets of the previous point.*

Proof. Given a rooted edge-decorated hypertree H with k edges and n vertices and with root labelled by r, the edges give a set $\mathbb{S}$ of k sets on $n-1$ vertices with an $\mathcal{S}$-structure on each of them. Indeed, considering an edge e, the set of vertices of e different from the petiole of e is endowed with an $\mathcal{S}'$-structure, because the rooted hypertree is edge-decorated. Moreover, every vertex different from the root is the petiole of all edges containing it, except the closest edge from the root, which always exists as the hypertree is connected. Therefore, every vertex different from the root is counted once in the sets of $\mathbb{S}$.

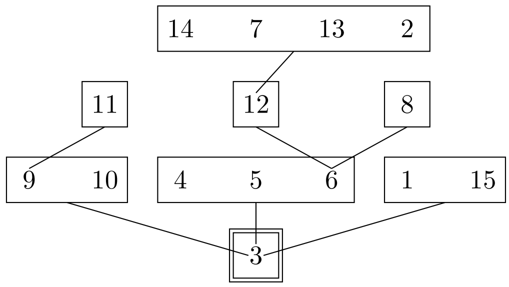
Then, considering the k previous sets and the root as the set of vertices V, for all edges e in H, we can link the corresponding vertex of V to the petiole of e which belongs to another vertex of V: let us call BT the result. As H is an hypertree, for every vertex v there is only one path between the root and v thus, there is also one and only one path from the root to the vertices containing v in BT: BT is a box tree with root labelled by r.

Conversely, given a box tree BT with root labelled by r and the other vertices labelled by one of the k sets on $n-1$ vertices with an $\mathcal{S}'$-structure on each of them, we can call *father* of a vertex the label linked to it with an edge. Calling the set

of labels V, we can define E as a set of subsets of V obtained by taking, for every vertex of BT, the union of the set of its labels with its father. We thus obtain an hypertree rooted in r. Moreover, the $\mathcal{S}$-structure on every vertex of BT induces an $\mathcal{S}$-edge-decoration of the rooted hypertree.

Let us remark that this application is a bijection of sets but not a bijection of species.

Example 2.5. The previous bijection associates to the edge-decorated rooted hypertree of the example 1.24 the root 3, the set of lists $\{(15,1), (14,7,13,2), (4,5,6), (8), (10,9), (11), (12)\}$ and the following box tree:



The elements of each box are not ordered.

□

Theorem 2.6. *Every hollow hypertree with k edges and n vertices, whose edges are decorated by $\mathcal{S}$ can be decomposed as a pair $(\mathbb{S}, BT)$ where:*

- *$\mathbb{S}$ is a set of k sets on n vertices with an $\mathcal{S}'$-structure on each of them,*
- *and BT is a box tree on k vertices with vertices labelled by one of the k sets of the previous point.*

Proof. We adapt the proof of the theorem for rooted hypertrees to hollow hypertrees.

Given a hollow edge-decorated hypertree H with k edges and n vertices, the edges give a set $\mathbb{S}$ of k sets on n vertices with an $\mathcal{S}$-structure on each of them. Indeed, considering an edge e, the set of vertices of e different from the petiole or the gap of e is endowed with an $\mathcal{S}'$-structure, because the hollow hypertree is edge-decorated. Moreover, every vertex is the petiole of all edges containing it, except the closest from the hollow edge, which always exists as the hypertree is connected. Therefore every vertex is counted exactly once in the set $\mathbb{S}$.

Then, considering the k previous sets as the set of vertices V, for all non-hollow edges e in H, we can link the corresponding vertex of V to the petiole of e which belongs to another vertex of V: let us call BT the result. The vertices that were in the hollow edge form the root of BT.As H is an hypertree, for every vertex v there is only one path between the hollow edge and v thus, there is also one and only one path from the root to the vertices containing v in BT: BT is a box tree with root labelled by vertices of the hollow edge.

Conversely, given a box tree BT with vertices labelled by one of the k sets on n vertices with an $\mathcal{S}'$-structure on each of them, we can call *father* of a vertex the label linked to it with an edge. Calling the set of labels V, we can define E as a set of subsets of V obtained by taking, for every vertex of BT different from its root, the union of the set of its labels with its father and by taking the set of labels of the root of BT with a gap. We thus obtain a hollow hypertree. Moreover, the $\mathcal{S}$-structure on every vertex of BT induces an $\mathcal{S}$-edge-decoration of the hollow hypertree.

Example 2.7. The example 2.2 is the box tree associated to the hypertree of the example 1.28, with the set of pointed sets $\{\{\mathbf{1},2,3\},\{6,\mathbf{7}\},\{\mathbf{8}\},\{\mathbf{9}\},\{\mathbf{4}\},\{\mathbf{5}\}\}$.

$\square$

To each species F, we can associate the following generating series:

$$C_F(x)=\sum_{n\geq 0}\#F(\{1,\ldots,n\})\frac{x^n}{n!}.$$

We can combine this series with the weight t.

Example 2.8. The generating series of species defined in the example 1.16 are:

- $C_{\mathrm{Assoc}}(x)=\frac{1}{1-x}$,
- $C_{\mathcal{E}}(x)=\exp(x)$,
- $C_X(x)=x$,
- $C_{\mathrm{Comm}}(x)=\exp(x)-1$.

Let $S_{\mathcal{S}}$, $S_{\mathcal{S}}^p$, $S_{\mathcal{S}}^c$, $S_{\mathcal{S}}^a$ and $S_{\mathcal{S}}^{pa}$ be the weighted generating series of $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{S}}^p$, $\mathcal{H}_{\mathcal{S}}^c$, $\mathcal{H}_{\mathcal{S}}^a$ and $\mathcal{H}_{\mathcal{S}}^{pa}$. Let also $E_{\mathcal{S}}(k,n)$ be the number of sets of k sets on n vertices with an $\mathcal{S}'$-structure on each of them.

Using proposition 2.3, the previous theorems 2.4 and 2.6 imply:

Corollary 2.9. *Given $\mathcal{S}$, a species, the generating series of the species of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees have the following expressions:*

$$S_{\mathcal{S}}^p(x)=\frac{x}{t}+\sum_{n\geq 2}\sum_{k=1}^{n-1}n\times E_{\mathcal{S}}(k,n-1)n^{k-1}t^{k-1}\frac{x^n}{n!}\tag{2.1}$$

and

$$S_{\mathcal{S}}^c(x)=x+\sum_{n\geq 2}\sum_{k=1}^{n}E_{\mathcal{S}}(k,n)n^{k-1}t^{k-1}\frac{x^n}{n!}.\tag{2.2}$$

Proof.

- According to theorem 2.4, the number of rooted hypertrees on k edges and n vertices is given by:
$$(S_{\mathcal{S}}^p)_{n,k}(x)=n\times E_{\mathcal{S}}(k,n-1)n^{k-1}t^{k-1},$$
where we have n different ways to choose the root, $E_{\mathcal{S}}(k,n-1)$ ways to make a set of k sets on the $n-1$ vertices left with an $\mathcal{S}'$-structure on each of them and $1\times n^{k-1}$ ways to organize these sets into a box tree with its root fixed, according to proposition 2.3.
- Let us apply theorem 2.6. We have $E_{\mathcal{S}}(k,n)$ manners to make a set of k sets on the n vertices with an $\mathcal{S}'$-structure on each of them. Then consider that $n_{i_1},\ldots,n_{i_k}$ are the cardinality of each of these sets. We choose the j-th of these sets as the root of a box tree on k vertices and n labels. The number of box trees obtained is $n_{i_j}\times n^{k-2}$, according to proposition 2.3. As $\sum_{j=1}^k n_{i_j}\times n^{k-2}=n^{k-1}$, we obtain the expected result.

$\square$

Example 2.10.

- Let Assoc be the species of non empty lists. The number of partitions of a set of cardinality n in k lists is $\binom{n-1}{k-1}\times\frac{n!}{k!}$. Therefore, if we consider hypertrees decorated by $\widehat{\mathrm{Assoc}}$, the generating series of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees are:
$$S_{\mathcal{S}}^p(x)=\frac{x}{t}+\sum_{n\geq 2}\sum_{k=1}^{n-1}n\times\binom{n-2}{k-1}\times\frac{(n-1)!}{k!}(nt)^{k-1}\frac{x^n}{n!}$$

and

$$S_{\mathcal{S}}^{c}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} \binom{n-1}{k-1} \times \frac{n!}{k!}(nt)^{k-1}\frac{x^n}{n!}.$$

- Let Perm be the species of non empty pointed sets. The number of partitions of a set of cardinality n in k pointed sets is $\binom{n}{k} \times k^{n-k}$. Indeed, we choose the pointed vertex in each set and then we map the other vertices to these pointed vertices. Therefore, if we consider hypertrees decorated by $\widehat{\mathrm{Perm}}$, the generating series of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees are:

$$S_{\mathcal{S}}^{p}(x) = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=1}^{n-1} n \times \binom{n-1}{k} \times k^{n-1-k}(nt)^{k-1}\frac{x^n}{n!}$$

and

$$S_{\mathcal{S}}^{c}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} \binom{n}{k} \times k^{n-k}(nt)^{k-1}\frac{x^n}{n!}.$$

- Let Comm be the species of non empty sets. The number of partitions of a set of cardinality n in k sets is given by $S(n,k)$, a Stirling number of the second type. Therefore, if we consider hypertrees decorated by $\widehat{\mathrm{Comm}}$, the generating series of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees are:

$$S_{\mathcal{S}}^{p}(x) = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=1}^{n-1} nS(n-1,k)(nt)^{k-1}\frac{x^n}{n!},$$

and

$$S_{\mathcal{S}}^{c}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} S(n,k)(nt)^{k-1}\frac{x^n}{n!}.$$

These decorated hypertrees are isomorphic to non-decorated hypertrees. This result was first proven by I. Gessel and L. Kalikow in [GK05].

- The number of partitions of a set of cardinality n in k cycles is given by $|s(n,k)|$, the absolute value of a Stirling number of the first kind. Therefore, if we consider hypertrees decorated by the species of cycles, the generating series of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees are:

$$S_{\mathcal{S}}^{p}(x) = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=1}^{n-1} n \times |s(n-1,k)|(nt)^{k-1}\frac{x^n}{n!}$$

and

$$S_{\mathcal{S}}^{c}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} |s(n,k)|(nt)^{k-1}\frac{x^n}{n!}.$$

These series are the same as the one for the decoration by $\widehat{\mathrm{Lie}}$. We will see this last case in details in section 3.2.

Thanks to the corollary 2.9 and to the proposition 1.37, we get the following corollary:

Corollary 2.11. *The generating series of the species of edge-decorated hypertrees is given by:*

$$S_{\mathcal{S}}(x) = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=1}^{n-1} E_{\mathcal{S}}(k,n-1)n^{k-1}t^{k-1}\frac{x^n}{n!}, \tag{2.3}$$

where $E_{\mathcal{S}}(k,n)$ is the number of sets of k sets on n vertices with an $\mathcal{S}'$-structure on each of them.

The generating series of the species of rooted edge-decorated-and-pointed hypertrees and the species of edge-decorated-and-pointed hypertrees are related to the species of rooted edge-decorated hypertrees and the species of hollow edge-decorated hypertrees by the following relations:

$$S_{\mathcal{S}}^{pa}(x) = tS_{\mathcal{S}}^{p}(x) \times S_{\mathcal{S}}^{c}(x), \tag{2.4}$$

$$S_{\mathcal{S}}^{a}(x) = S_{\mathcal{S}}(x) + S_{\mathcal{S}}^{pa}(x) - S_{\mathcal{S}}^{p}(x). \tag{2.5}$$

These formulas give the cardinality of all types of decorated hypertrees in terms of decorated sets.

Proof. We prove the equation (2.3) by an integration of the formula of corollary 2.9. The other equations are obtained by translating in terms of generating series the equations (1.6) and (1.7) on species. □

2.3. **Refinement.** Let us now introduce two weights: one on the set of box trees and the other on the set of rooted and hollow hypertrees.

Definition 2.12. Let $BT = (L, V, R, E)$ be a box tree, we define the following weight on it:

$$W(BT) = \prod_{i \in L} x_i^{\#E^{-1}(i)}.$$

The power of x_i is then the number of sons of the label i, for every label i of L.

With this weight, the number of box trees is given by:

Proposition 2.13. *The number of box trees $N_{\{i_1,\ldots,i_p\};k,n}$ with n labels $\{i_1,\ldots,i_n\}$, $k+1$ vertices, k edges and its root labelled by $\{i_1,\ldots,i_p\}$ is:*

$$N_{\{i_1,\ldots,i_p\};k,n} = (x_{i_1} + \ldots + x_{i_p}) \times (\sum_{i=1}^{n} x_i)^{k-1}.$$

Proof. We prove this statement by induction on r. Let us call V_1 the root and $V_2, \ldots, V_k$ the other vertices.

For $r = 1$, there is only one vertex attached to a label of the root. Thus, there is $\sum_{i \in V_1} x_i$ box trees on two vertices with root V_1.

If this statement holds for all $k < r$, we want the number $N_{\{i_1,\ldots,i_p\};r,n}$ of box trees on $r+1$ vertices. It can be obtained by summing on the number of vertices attached to the root, called its *sons*.

If the root has j sons $V_{a_1}, \ldots V_{a_j}$, each of them is attached to a label i_f of the root: this gives a term $(x_{i_1} + \ldots + x_{i_p})^j$. Moreover, cutting the root and linking together the sons of the root, we obtain a box tree on $r+1-j$ vertices and with the same labels except the ones of V_0 which is deleted, with root having the labels of $V_{a_1} \cup \ldots \cup V_{a_j}$. Using the induction hypothesis, we obtain:

$$N_{\{i_1,\ldots,i_p\};k,n} = \sum_{j=1}^{r-1} (x_{i_1} + \ldots + x_{i_p})^j \times \sum_{1 < i_1 < \ldots < i_j} \left(\sum_{i \in V_{i_1} \cup \ldots \cup V_{i_j}} x_i \right) \left(\sum_{i \notin V_1} x_i \right)^{r-j-1}.$$

In the second sum $\sum_{1<i_1<\ldots<i_j} (\sum_{i \in V_{i_1} \cup \ldots \cup V_{i_j}} x_i)$, every $x_\alpha \in V_i$ appears $\binom{r-1}{j-1}$ times, for $i \geq 2$. Therefore, we have:

$$N_{\{i_1,\dots,i_p\};k,n} = \sum_{j=1}^{r-1} \binom{r-1}{j-1} (\sum_{i\in V_1} x_i)^j \times (\sum_{i\notin V_1} x_i)^{r-j} = (\sum_{i\in V_1} x_i) \times (\sum_{l=1}^{p} \sum_{i\in V_l} x_i)^{r-1}.$$

□

The weight on the set of hypertrees on n vertices is defined as follow.

Definition 2.14. Let $H = (V, E)$ be a rooted or a hollow hypertree on n vertices, we define the following weight on it:

$$W(H) = \prod_{i\in V} x_i^{p(i)},$$

where $p(i)$ is the number of times when i is the petiole of an edge.

These weights are related by the following proposition:

Theorem 2.15. *Given a species $\mathcal{S}$, every weighted rooted hypertree on n vertices with k edges, whose edges are decorated by $\mathcal{S}$ can be decomposed as a triple $(r, \mathbb{S}, BT)$ where:*

- *r is the root of the hypertree,*
- *$\mathbb{S}$ is a set of k sets on $n-1$ vertices with an $\mathcal{S}'$-structure on each of them,*
- *and BT is a weighted box tree on $k+1$ vertices with root labelled by $\{i_1, \dots, i_p\}$ and the other vertices labelled by $\{i_{p+1}, \dots, i_n\}$.*

Proof. We only have to prove the compatibility with weights of the bijection of the proof of theorem 2.4. This compatibility is due to the fact that the set of petioles of both rooted hypertree and box tree related in the theorem is the same.

Example 2.16. The weight of the rooted hypertree of the example 1.24, which corresponds to the weight of the box tree of the example 2.5, is: $x_3^3 x_6^2 x_9 x_{12}$.

□

Theorem 2.17. *Every weighted hollow hypertree on n vertices with k edges, whose edges are decorated by $\mathcal{S}$ can be decomposed as a pair $(\mathbb{S}, BT)$ where:*

- *$\mathbb{S}$ is a set of k sets on $n-1$ vertices with an $\mathcal{S}'$-structure on each of them,*
- *and BT is a weighted box tree on k vertices with vertices labelled by $i_1 \dots i_n$.*

Proof. In the same way as we proved the compatibility with weights of the bijection of the proof of theorem 2.4 in the case of rooted weighted hypertrees, we can prove that the proof of theorem 2.6 is compatible with weights.

Example 2.18. The weight of the rooted hypertree of the example 1.28, which corresponds to the weight of the box tree of the example 2.2, is: $x_2 x_3 x_4 x_7^2$.

□

Let $S_{\mathcal{S},W}$, $S^p_{\mathcal{S},W}$, $S^c_{\mathcal{S},W}$, $S^a_{\mathcal{S},W}$ and $S^{pa}_{\mathcal{S},W}$ be the weighted generating series of $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}^p_{\mathcal{S}}$, $\mathcal{H}^c_{\mathcal{S}}$, $\mathcal{H}^a_{\mathcal{S}}$ and $\mathcal{H}^{pa}_{\mathcal{S}}$. Let also be $E_{\mathcal{S}}(k,n)$ the number of sets of k sets on $n-1$ vertices with an $\mathcal{S}'$-structure on each of them.

Using proposition 2.13, the previous theorems 2.17 and 2.15 imply:

Corollary 2.19. *The generating series of the species of edge-decorated rooted hypertrees and edge-decorated hollow hypertrees have the following expressions:*

$$S^p_{\mathcal{S}}(x) = \frac{x}{t} + \sum_{n\geq 2} \sum_{k=1}^{n-1} n \times E_{\mathcal{S}}(k, n-1)(x_1 + \ldots + x_n)^{k-1} t^{k-1} \frac{x^n}{n!} \tag{2.6}$$

and

$$S^c_{\mathcal{S}}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} E_{\mathcal{S}}(k,n)(x_1+\ldots+x_n)^{k-1}t^{k-1}\frac{x^n}{n!}. \tag{2.7}$$

This corollary will be used in the next section where we specialize the results for operads Lie and PreLie.

3. Two cases of edge-decorated hypertrees

3.1. $\widehat{\mathrm{PreLie}}$**-decorated hypertrees.** In their article [JMM07b], C. Jensen, J. McCammond and J. Meier introduce a weight on the set of hypertrees. This weight seems to be related to a decoration of edges by the linear species $\widehat{\mathrm{PreLie}}$. Therefore, in this part, we consider hypertrees decorated by the linear species $\widehat{\mathrm{PreLie}}$ whose differential is PreLie, which associates to a finite set I the set of labelled rooted trees with labels in I.

3.1.1. *Application of the enumeration of decorated hypertrees.* Applying the results of the previous section on counting edge-decorated hypertrees, we obtain the following proposition:

Proposition 3.1. *The generating series of the species of rooted hypertrees decorated by the linear species* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^p_{\widehat{\mathrm{PreLie}}} = \frac{x}{t} + \sum_{n\geq 2} n(tn+n-1)^{n-2}\frac{x^n}{n!}.$$

The generating series of the species of hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S_{\widehat{\mathrm{PreLie}}} = x + \sum_{n\geq 2} (tn+n-1)^{n-2}\frac{x^n}{n!}.$$

The generating series of the species of hollow hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S^c_{\widehat{\mathrm{PreLie}}} = x + \sum_{n\geq 2} (tn+n)^{n-1}\frac{x^n}{n!}.$$

Proof. There is a classical formula for the number of forest of k trees on n vertices, which can be found in the book of M. Aigner and G. Ziegler [AZ04]:

$$E_{\mathrm{PreLie}}(k,n) = \binom{n}{k} k \times n^{n-1-k} = \binom{n-1}{k-1} \times n^{n-k}. \tag{3.1}$$

Using the expressions of the corollaries 2.9 and 2.11, we obtain:

$$S^p_{\widehat{\mathrm{PreLie}}}(x) = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=1}^{n-1} n \times \binom{n-2}{k-1} \times (n-1)^{n-1-k}(nt)^{k-1}\frac{x^n}{n!}$$

and

$$S^c_{\widehat{\mathrm{PreLie}}}(x) = x + \sum_{n\geq 2}\sum_{k=1}^{n} n \times \binom{n-1}{k-1} \times n^{n-1-(k-1)} \times (nt)^{k-1}\frac{x^n}{n!}.$$

Re-indexing the sums, it gives:

$$S^p_{\widehat{\mathrm{PreLie}}} = \frac{x}{t} + \sum_{n\geq 2}\sum_{k=0}^{n-2} n \times \binom{n-2}{k} \times (n-1)^{n-2-k}(nt)^{k}\frac{x^n}{n!}$$

and

$$S^c_{\widehat{\mathrm{PreLie}}}(x) = x + \sum_{n\geq 2}\sum_{k=0}^{n-1} \binom{n-1}{k} \times n^{n-1-k} \times (nt)^{k}\frac{x^n}{n!}.$$

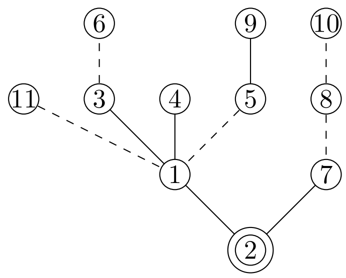
Using the binomial theorem, we obtain the expected results for $S^p_{\widehat{\mathrm{PreLie}}}$ and $S^c_{\widehat{\mathrm{PreLie}}}$. The series $S_{\widehat{\mathrm{PreLie}}}$ is then obtained by the use of the first equation of proposition 1.37.

□

3.1.2. *Link with 2-coloured rooted trees.* We now draw the link between trees whose edges can be either blue (0) or red (1) and edge-decorated hollow hypertrees.

Definition 3.2. A *2-coloured rooted tree* is a rooted tree (V, E), where V is the set of vertices and $E \subseteq V \times V$ is the set of edges, together with a map φ from E to $\{0, 1\}$. It is equivalent to the data of a tree (V, E) and a decomposition $E_0 \cup E_1$ of E, with $E_0 \cap E_1 = \emptyset$.

Example 3.3. An example of 2-coloured rooted tree. The edges of E_1 are dashed whereas the edges of E_0 are plain.



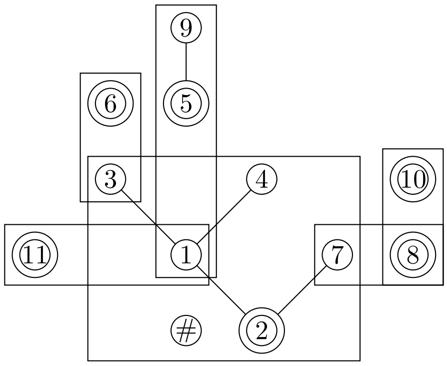
Proposition 3.4. *The species of hollow hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is isomorphic to the species of 2-coloured rooted trees.*

Proof. A hollow hypertree with edges decorated by $\widehat{\mathrm{PreLie}}$ is a hollow tree in which, for all edges e, the vertices of e different from the gap or the petiole form a rooted tree.

Let us consider a hollow hypertree H decorated by $\widehat{\mathrm{PreLie}}$ on vertex set V. We call E_0, the set of edges between elements of V in the rooted trees obtained from the decoration by $\widehat{\mathrm{PreLie}}$. The graph (V, E_0) is then a forest of trees obtained by deleting the edges of the hypertree H and forgetting the roots. For any edge e of H, we write r_e for the root of the rooted tree in e and p_e for the petiole of e. Let E_1 be the set of edges between r_e and p_e for all edges e of H. By definition of the sets, the intersection of E_0 with E_1 is empty. Moreover, to every path in H corresponds a path in $(V, E_0 \cup E_1)$. As H is an hypertree, the graph $(V, E_0 \cup E_1)$ is a tree T. We root that tree in the root r of the tree in the hollow edge of H: T is then a 2-coloured tree rooted in r.

Conversely, let $T = (V, E_0 \cup E_1)$ be a 2-coloured rooted tree. The graph (V, E_0) is a forest of trees: we can root these trees in their closest vertex from the root. Let us call $T_1, \ldots, T_n$ this forest, where T_1 is the tree rooted in the root of T. For all i between 2 and n, there is one vertex of V closer than the root of T_i from the root of T: we call this vertex p_i. Then, we consider the hypergraph whose set of vertices is V, with edges containing the vertices of T_1 or the vertices of a T_i and p_i for all i between 2 and n. Adding the edges of every T_i, we obtain an hypergraph decorated by $\widehat{\mathrm{PreLie}}$. Moreover, paths in T and in the hypergraph are the same: the hypergraph is then an hypertree.

Example 3.5. The hollow tree with edges decorated by $\widehat{\mathrm{PreLie}}$ associated to the 2-coloured rooted tree of the example 3.3 is below.



□

Remark that this proposition gives the expression for $S^c_{\widehat{\text{PreLie}}}$ found previously in the proposition 3.1.

Let us now link rooted edge-pointed hypertrees decorated by $\widehat{\text{PreLie}}$ with 2-coloured trees. We consider that edges of E_0 are blue and edges of E_1 are red.

Proposition 3.6. *The set of rooted edge-pointed hypertrees decorated by* $\widehat{\text{PreLie}}$ *is in one-to-one correspondence with the set of 2-coloured rooted trees whose root has exactly one blue son (and possibly some red sons).*

Proof. Given a rooted edge-pointed hypertrees decorated by $\widehat{\text{PreLie}}$ H with k edges and n vertices, we consider its pointed set of vertices (r, V). We draw in blue all the edges obtained from the PreLie-structure in the edges of H: it gives a forest of $k+1$ trees with $n-k-1$ blue edges. We link r with the root of the tree in the pointed edge of H with a blue edge: it gives a forest of k trees with $n-k$ blue edges. Now, for all non-pointed edges e, we link the petiole of e with the root of the tree in e. We thus obtain a 2-coloured graph G on n vertices with $n-k+k-1$ edges. Let us show that this graph is connected. A path between the root and a vertex x in H is a sequence of paths in trees of the edges of H from the root to a vertex which is the petiole of an edge and paths from the petiole of an edge to the root of the corresponding trees. As these paths also exist in G, we obtain a 2-coloured rooted trees whose root has exactly one blue son and possibly some red sons.

Conversely, given such a 2-coloured tree T, we delete red edges of T and we root all connected component in the vertices that was the closest in T from the root of T. For every connected component C_k, we put C_k and the father of the root of C_k in the edge of an hypergraph H. We root the hypergraph in the root of T, point the edge containing the blue edge adjacent to the root and delete this blue edge. We thus obtain a rooted edge-pointed hypertrees decorated by $\widehat{\text{PreLie}}$. Indeed, the hypergraph is rooted and edge-pointed and every edge contains a rooted tree and a vertex so that the hypergraph can be seen as decorated by $\widehat{\text{PreLie}}$. Moreover, every vertex connected by an edge in T are in the same edge in H so H is connected. Finally, a cycle in H would come from a cycle in T, which does not exist so H is an hypertree. The weight of the hypertree corresponds to the number of red edges. □

3.1.3. *Computation of generating series of edge-pointed hypertrees.* We now use the analogy with 2-coloured rooted trees and the corollary 2.11 to compute the value of the other generating series:

Proposition 3.7. *The generating series of the species of rooted edge-pointed hypertrees decorated by* $\widehat{\text{PreLie}}$ *is given by:*

$$S^{pa}_{\widehat{\text{PreLie}}} = x + \sum_{n\geq 2} n(n+tn-1)^{n-3}(n-1)(1+2t)\frac{x^n}{n!}.$$

The generating series of the species of edge-pointed hypertrees decorated by $\widehat{\text{PreLie}}$ *is given by:*

$$S^{a}_{\widehat{\text{PreLie}}} = x + \sum_{n\geq 2} (n+tn-1)^{n-3}(n-1)(1+tn)\frac{x^n}{n!}.$$

Proof.

- We linked rooted edge-pointed hypertrees decorated by $\widehat{\text{PreLie}}$ with 2-coloured rooted trees whose root has exactly one blue son (and possibly some red sons) in the proposition 3.6. We count the 2-coloured rooted trees using as a weight on 2-coloured trees the number of red edges.

  Then, there is n ways to choose the root of the 2-coloured rooted tree. If the root have k sons, $k-1$ of them are red, there is k ways to choose the blue one and the set of the sons forms a forest of k 2-coloured rooted trees on $n-1$ vertices.

  Let us count the forests of k 2-coloured rooted trees on $n-1$ vertices. We use the equation (3.1):

  $$E_{\text{PreLie}}(k, n-1) = \binom{n-2}{k-1} \times (n-1)^{n-1-k}.$$

  There is $n-1-k$ edges in the forest and each of them is either blue or red. Hence the number of forests of k 2-coloured rooted trees on $n-1$ vertices is $\binom{n-2}{k-1} \times (n-1)^{n-1-k}(1+t)^{n-1-k}$.

  Finally, the n-th coefficient of the series is:

  $$(S^{pa}_{\widehat{\text{PreLie}}})_n = n\sum_{k=1}^{n-1} t^{k-1} \times \binom{n-2}{k-1} \times ((n-1)(1+t))^{n-1-k} \times k.$$

  A quick computation using re-indexation and the binomial theorem gives the result.
- We obtain the second generating series thanks to the dissymetry principle of proposition 1.35.

□

3.1.4. *Refinement.* We now use weights defined in part 2.3 on rooted and hollow hypertrees.

Proposition 3.8. *The generating series of the weighted species of hollow hypertrees decorated by* $\widehat{\text{PreLie}}$ *is given by:*

$$S^{c}_{\widehat{\text{PreLie}}} = x + \sum_{n\geq 2} ((x_1+\ldots+x_n)t+n)^{n-1}\frac{x^n}{n!}.$$

The generating series of the weighted species of rooted hypertrees decorated by $\widehat{\text{PreLie}}$ *is given by:*

$$S^{p}_{\widehat{\text{PreLie}}} = x + \sum_{n\geq 2} n((x_1+\ldots+x_n)t+n-1)^{n-2}\frac{x^n}{n!}.$$

Proof. We use the formulas of 2.17 and 2.15. □

We can separate the second n into $y_1, \ldots, y_n$ by introducing the following weight on PreLie-decorated hollow hypertrees. Therefore, we need some definitions:

Definition 3.9. Given a rooted forest and a vertex v of it, a *child* v' of the vertex v is a vertex v' linked to v by an edge and such that v is on the path from v' to the root. The degree of v in the rooted forest is the number of children of v in it.

Using this definition, we usually define the following weight on the set of forests of rooted trees on a set of vertices V.

Definition 3.10. Let F be a forest of rooted trees on a set of vertices V, we define the following weight on it:

$$W(F) = \prod_{i \in V} y_i^{s(i)},$$

where $s(i)$ is the number of children of the vertex i.

The bijection between $\widehat{\mathrm{PreLie}}$-decorated hollow hypertrees and box trees and set of decorated sets of the proposition 2.6 and the weights defined in 2.12 and 3.10 give a weight on the set of $\widehat{\mathrm{PreLie}}$-decorated hollow hypertrees:

Definition 3.11. Let H be a $\widehat{\mathrm{PreLie}}$-decorated hollow hypertree on a set of vertices V, we define the following weight on it:

$$W(T) = \prod_{i \in V} x_i^{p(i)} y_i^{s(i)},$$

where $p(i)$ is the number of edges of H whose petiole is i and $s(i)$ is the number of child of v in the tree of the $\widehat{\mathrm{PreLie}}$ decoration.

Using the bijection with box trees and sets of decorated sets, we can compute the generating series of the weighted species of $\widehat{\mathrm{PreLie}}$-decorated hollow hypertrees.

Proposition 3.12. *The generating series of the weighted species of hollow hypertrees whose edges are decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$S_t^c = x + \sum_{n \geq 2} ((x_1 + \ldots + x_n)t + y_1 + \ldots y_n)^{n-1} \frac{x^n}{n!}.$$

Proof. In the book of Stanley [Sta01, proposition 5.3.2], the formula for the number of forest of k trees on n vertices is given by:

$$E_{\mathrm{PreLie}}(t, k, n) = \binom{n-1}{k-1} (\sum_{j=1}^{n} \sum_{i \in V_j} y_i)^{n-k}. \tag{3.2}$$

We use the corollary 2.19 to obtain the expected results. □

3.1.5. *Computation of cycle index series of hypertrees decorated by* $\widehat{\mathrm{PreLie}}$. The reader may consult the appendix 5 for basic definitions on cycle index series. We denote the cycle index series of usual species in the same way that the species itself. We compute the cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$. We do not write the argument of the cycle index series $(t, p_1, p_2, \ldots)$ in this subsection.

Proposition 3.13. *The cycle index series of hollow hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^c_{\widehat{\mathrm{PreLie}}} = \frac{1}{1+t} \mathrm{PreLie} \circ (1+t) p_1. \tag{3.3}$$

Proof. By proposition 3.4, the cycle index series of hollow hypertrees decorated by $\widehat{\mathrm{PreLie}}$ is given by the cycle index series of 2-coloured rooted trees. □

Let us define the following expressions, for λ a partition of an integer n, written $\lambda \vdash n$:

$$f_k(\lambda) = \sum_{l|k} l\lambda_l$$

and

$$P_k(\lambda) = \left(\left((1+t^k)f_k(\lambda)-1\right)^{\lambda_k} - k\lambda_k(t^k+1) \times \left((1+t^k)f_k(\lambda)-1\right)^{\lambda_k-1}\right).$$

We obtain the following expression for the cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$:

Proposition 3.14. *The cycle index series of rooted hypertrees decorated by* $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^p_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n, \lambda_1\neq 0} \lambda_1(\lambda_1 t+\lambda_1-1)^{\lambda_1-2}\prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{3.4}$$

The cycle index series of edge-pointed rooted hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^{pa}_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n, \lambda_1\neq 0} \lambda_1(\lambda_1-1)(2t+1)(\lambda_1+\lambda_1 t-1)^{\lambda_1-3}\prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{3.5}$$

The cycle index series of edge-pointed hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z^{a}_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n, \lambda_1\neq 0} (\lambda_1-1)(1+\lambda_1 t)(\lambda_1+\lambda_1 t-1)^{\lambda_1-3}\prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{3.6}$$

The cycle index series of hypertrees decorated by $\widehat{\mathrm{PreLie}}$ *is given by:*

$$Z_{\widehat{\mathrm{PreLie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n, \lambda_1\neq 0} (\lambda_1 t+\lambda_1-1)^{\lambda_1-2}\prod_{k\geq 2} P_k(\lambda) \times \frac{p_\lambda}{z_\lambda}. \tag{3.7}$$

Proof. We use the proposition 1.37 and the cycle index series of hollow hypertrees decorated by $\widehat{\mathrm{PreLie}}$ to compute the other series.

- The series $Z^p_{\widehat{\mathrm{PreLie}}}$ satisfies:

$$Z^p_{\widehat{\mathrm{PreLie}}} = \frac{p_1}{t} \times (1+\mathrm{Comm}) \circ (t \times Z^c_{\widehat{\mathrm{PreLie}}}).$$

  Using the first step of the proof, it gives:

$$Z^p_{\widehat{\mathrm{PreLie}}} = \frac{p_1}{t} \times (1+\mathrm{Comm}) \circ (\frac{t}{1+t}\,\mathrm{PreLie} \circ ((1+t)p_1)).$$

  We now use methods from the proof of the lemma 4 in the article of V. Dotsenko and A. Khoroshkin [DK07] and from the proof of the proposition 7.2 in the article of F. Chapoton [Cha07].

  Consider $\lambda = (\lambda_1+1, \ldots, \lambda_r)$. We can suppose that, for $i \geq r$, we have $p_i = 0$. The coefficient of tp_λ in $Z^p_{\widehat{\mathrm{PreLie}}}$ is given by the multiple residue:

$$c_\lambda = \int (1+\mathrm{Comm}) \circ \frac{t}{1+t}\,\mathrm{PreLie} \circ ((1+t)p_1) \prod_{i=1}^{r} \frac{dp_i}{p_i^{\lambda_i+1}},$$

  with $1+\mathrm{Comm} = \exp(\sum_{k\geq 1}\frac{p_k}{k})$.

We use the substitution $y_l = p_l \circ \frac{t}{1+t}\,\mathrm{PreLie} \circ ((1+t)p_1)$:

$$c_\lambda = \int \exp(\sum_{k\geq 1} \frac{y_k}{k}) \prod_{k=1}^{r} \exp(\lambda_k \sum_{l\geq 1} \frac{y_{kl}}{l} \times \frac{1+t^{kl}}{t^{kl}}) \times (t^k)^{\lambda_k} \times (1 - y_k \frac{1+t^k}{t^k}) \prod_{i=1}^{r} \frac{dy_i}{y_i^{\lambda_i+1}}.$$

Yet the argument of exp can be rewritten as:

$$\sum_{k\geq 1} k\lambda_k \sum_{l\geq 1} \frac{y_{kl}}{kl} \times \frac{1+t^{kl}}{t^{kl}} = \sum_{k\geq 1} \frac{y_k}{k} \times \frac{1+t^k}{t^k} \left(f_k(\lambda) - 1\right),$$

with $f_k(\lambda) = \left(\sum_{l|k, l>1} l\lambda_l\right) + \lambda_1 + 1$.

Therefore, c_λ is given by the residue:

$$c_\lambda = \prod_{k=1}^{r} \int \exp\left(\frac{y_k}{k} \times \left(1 + \frac{1+t^k}{t^k}\left(f_k(\lambda) - 1\right)\right)\right) t^{k\lambda_k} (1 - y_k \frac{1+t^k}{t^k}) \frac{dy_k}{y_k^{\lambda_k+1}}.$$

This gives the expected result as the residue of $\exp(az)z^{-n}$, for a constant a, is given by $\frac{a^{n-1}}{(n-1)!}$.

- The relation (1.7) gives the following relation for the series $Z^{pa}_{\widehat{\mathrm{PreLie}}}$:

$$Z^{pa}_{\widehat{\mathrm{PreLie}}} = p_1 \times \left((p_1(1 + \mathrm{Comm})) \circ \frac{t}{1+t}\,\mathrm{PreLie} \circ ((1+t)p_1) \right).$$

We use the same method as for rooted hypertrees with the same substitution to obtain the result.

- The relation (1.6) gives the following relation for the series $Z^{a}_{\widehat{\mathrm{PreLie}}}$:

$$Z^{a}_{\widehat{\mathrm{PreLie}}} = (\widehat{\mathrm{PreLie}} - p_1) \circ tZ^{p}_{\widehat{\mathrm{PreLie}}}.$$

However the relation (50) in the article of F. Chapoton [Cha05] gives:

$$\widehat{\mathrm{PreLie}} = p_1 \left(1 + \mathrm{PreLie} + \frac{1}{\mathrm{PreLie}}\right).$$

Hence the series $Z^{a}_{\widehat{\mathrm{PreLie}}}$ satisfies:

$$Z^{a}_{\widehat{\mathrm{PreLie}}} = tZ^{p}_{\widehat{\mathrm{PreLie}}} \left(\mathrm{PreLie} \circ tZ^{p}_{\widehat{\mathrm{PreLie}}} + \frac{1}{\mathrm{PreLie} \circ tZ^{p}_{\widehat{\mathrm{PreLie}}}} \right).$$

Moreover, using the expression of $Z^{c}_{\widehat{\mathrm{PreLie}}}$ in terms of $Z^{p}_{\widehat{\mathrm{PreLie}}}$ of relation (1.5), and the one of the proposition, we obtain:

$$Z^{a}_{\widehat{\mathrm{PreLie}}} = p_1 \times \left((1 + \mathrm{Comm})(\frac{p_1}{t} + \frac{t}{p_1}) \right) \circ \frac{t}{1+t}\,\mathrm{PreLie} \circ ((1+t)p_1).$$

We use the same method as for rooted hypertrees with the same substitution to obtain the result.

- The series $Z_{\widehat{\mathrm{PreLie}}}$ is obtained by using the dissymetry principle 1.35.

□

3.2. $\widehat{\mathrm{Lie}}$**-decorated hypertrees.** In this part, we study $\widehat{\mathrm{Lie}}$-decorated hypertrees, where Lie is the species associated to the following cycle index series:

$$\mathrm{Lie} = \sum_{k\geq 1} \frac{\mu(k)}{k} \log(1 - p_k).$$

3.2.1. *Computation of generating series of hypertrees decorated by* $\widehat{\mathrm{Lie}}$. Applying the results of the second section, we obtain the following proposition:

Proposition 3.15. *The generating series of the species of rooted hypertrees decorated by* $\widehat{\mathrm{Lie}}$ *is given by:*

$$S^p_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 1} \frac{1}{t} \prod_{k=0}^{n-2} (nt+k) \times \frac{x^n}{n!}.$$

The generating series of the species of hypertrees decorated by $\widehat{\mathrm{Lie}}$ *is given by:*

$$S_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 1} \prod_{k=1}^{n-2} (nt+k) \times \frac{x^n}{n!}.$$

The generating series of the species of hollow hypertrees decorated by $\widehat{\mathrm{Lie}}$ *is given by:*

$$S^c_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 1} \prod_{k=1}^{n-1} (nt+k) \frac{x^n}{n!}.$$

The generating series of the species of rooted edge-pointed hypertrees decorated by $\widehat{\mathrm{Lie}}$ *is given by:*

$$S^{pa}_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 2} \sum_{p=1}^{n-1} \binom{n}{p} \prod_{k=0}^{n-2} \prod_{l=1}^{n-p-1} (pt+k)((n-p)t+l) \frac{x^n}{n!}.$$

The generating series of the species of edge-pointed hypertrees decorated by $\widehat{\mathrm{Lie}}$ *is given by:*

$$S^a_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 2} \prod_{k=0}^{n-2} (nt+k) - \sum_{p=0}^{n-1} \binom{n}{p} \prod_{k=0}^{n-2} \prod_{l=1}^{n-p-1} (pt+k)((n-p)t+l) \frac{x^n}{n!}.$$

Proof. The three first results are obtained by applying the expressions of corollary 2.9 and the formula (2.3) of the corollary 2.11 with $E_{\widehat{\mathrm{Lie}}}(k,n) = (-1)^{k,n} s(k,n)$, where $s(k,n)$ is a Stirling number of the first kind. Indeed, the Stirling numbers of the first kind satisfy the following classical equation:

$$\sum_{k=1}^{n} s(k,n) x^k = \prod_{k=0}^{n-1} (x-k).$$

The fourth one is obtained using the relation (1.7) and the last one is obtained by using the dissymetry principle of the proposition 1.35. □

3.2.2. *Computation of cycle index series of hypertrees decorated by* $\widehat{\mathrm{Lie}}$. Let us now study the action of the symmetric group on these hypertrees.

Let us define the following expressions, for λ a partition of an integer n:

$$f_k(\lambda) = \sum_{l|k} l\lambda_l$$

and

$$\varphi_i(\lambda) = \sum_{k|i} \frac{t^k}{i} \mu(\frac{i}{k}) f_k(\lambda),$$

where μ is the Möbius function.

Using these expressions, we have the following relations:

Proposition 3.16. *The cycle index series of hollow hypertrees decorated by* $\widehat{\mathrm{Lie}}$ *is given by:*

$$Z^c_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n}\sum_{p=1}^{r}\frac{\mu(p)}{p}\prod_{i=1,i\neq p}^{r}\left(\binom{\varphi_i(\lambda)+\lambda_i-1}{\lambda_i} - t^i\binom{\varphi_i(\lambda)+\lambda_i-1}{\lambda_i-1}\right)$$
$$\times\left(\sum_{q=1}^{\lambda_p}\frac{1}{q}\left(\binom{\varphi_p(\lambda)+\lambda_p-q-1}{\lambda_p-q} - t^q\binom{\varphi_p(\lambda)+\lambda_p-q-1}{\lambda_p-q-1}\right)\right)\times\frac{p_\lambda}{z_\lambda}.$$

The cycle index series of rooted hypertrees decorated by $\widehat{\mathrm{Lie}}$ *is given by:*
(3.8)
$$Z^p_{\widehat{\mathrm{Lie}}} = \sum_{n\geq 1}\sum_{\lambda\vdash n}\frac{1}{t}\prod_{k=0}^{\lambda_1-2}(\lambda_1 t+k)\times\left(\binom{\varphi_i(\lambda)+\lambda_i-1}{\lambda_i} - t^i\binom{\varphi_i(\lambda)+\lambda_i-1}{\lambda_i-1}\right)\times\frac{p_\lambda}{z_\lambda}.$$

Proof.
- The equations of corollary 1.38 give:

$$tZ^p_{\widehat{\mathrm{Lie}}} = p_1 + p_1\times \mathrm{Comm}\circ\left(t\times \mathrm{Lie}\circ tZ^p_{\widehat{\mathrm{Lie}}}\right)$$

and

$$Z^c_{\widehat{\mathrm{Lie}}} = \mathrm{Lie}\circ Z^p_{\widehat{\mathrm{Lie}}}.$$

Remark that in the last formula it is written Lie and not $(\mathrm{Lie}-1)$. This is because we already took Lie such that we have $\mathrm{Lie}(0,0,\ldots)=0$ by definition.

Consider $\lambda=(\lambda_1,\ldots,\lambda_r)$. We can suppose that, for $i\geq r$, we have $p_i=0$. The coefficient of tp_λ in $Z^p_{\widehat{\mathrm{Lie}}}$ is given by the residue c_λ:

$$c_\lambda = \int tZ^p_{\widehat{\mathrm{Lie}}}\prod_{i=1}^{r}\frac{dp_i}{p_i^{\lambda_i+1}}.$$

We use the substitution $y_i = p_i\circ tZ^p_{\widehat{\mathrm{Lie}}}$. Let us compute first $(1+\mathrm{Comm})\circ t\,\mathrm{Lie}$.

$$\begin{aligned}(1+\mathrm{Comm})\circ t\,\mathrm{Lie} &= \exp\left(\sum_{k\geq 1}\frac{t^k}{k}\times p_k\circ\left(-\sum_{l\geq 1}\frac{\mu(l)}{l}\log(1-p_l)\right)\right)\\ &= \exp\left(-\sum_{k\geq 1}\sum_{l\geq 1}\frac{t^k}{kl}\mu(l)\log(1-p_{kl})\right)\\ &= \prod_{k,l\geq 1}(1-p_{kl})^{-\frac{t^k\mu(l)}{kl}}.\end{aligned}$$

Hence the substitution is given by:

$$p_i = y_i\prod_{k,l\geq 1}(1-y_{kli})^{\frac{t^{ki}\mu(l)}{kl}}.$$

With this substitution, we obtain:

$$c_\lambda = \int y_1\prod_{i=1}^{r}\prod_{k,l\geq 1}(1-y_{kli})^{-\frac{\lambda_i t^{ki}\mu(l)}{kl}}\left(1-\frac{t^i y_i}{1-y_i}\right)\frac{dy_i}{y_i^{\lambda_i+1}}.$$

We then separate the terms in each of the y_j. It gives:

$$c_\lambda = \int y_1(1-y_1)^{-\lambda_1 t}\left(1-\frac{ty_1}{1-y_1}\right)\frac{dy_1}{y_1^{\lambda_1+1}}$$
$$\times \prod_{i=2}^{r}\int (1-y_i)^{-\varphi_i(\lambda)}\left(1-\frac{t^i y_i}{1-y_i}\right)\frac{dy_i}{y_i^{\lambda_i+1}},$$

where we set:

$$\varphi_i(\lambda) = \sum_{j|k|i}\frac{t^k}{i}\mu\left(\frac{i}{k}\right)j\lambda_j = \sum_{k|i}\frac{t^k}{i}\mu(\frac{i}{k})f_k(\lambda).$$

We now compute the following integral:

$$\int y_p^q(1-y_p)^{-\varphi_p(\lambda)-1}(1-y_p(1+t^p))\frac{dy_p}{y_p^{\lambda_p+1}}$$
$$= \sum_{j\geq 0}\binom{\varphi_p(\lambda)+\lambda_p-q}{\lambda_p-q}\left(\int y_p^{q+j-\lambda_p-1}dy_p - (1+t^p)\int y_p^{q+j-\lambda_p}dy_p\right)$$
$$= \binom{\varphi_p(\lambda)+\lambda_p-q-1}{\lambda_p-q} - t^p\binom{\varphi_p(\lambda)+\lambda_p-q-1}{\lambda_p-q-1}.$$

This computation applied to c_λ give the expected result for $Z^p_{\widehat{\mathrm{Lie}}}$.

- Calling $\lambda = (\lambda_1,\ldots,\lambda_r)$. We can suppose that, for $i\geq r$, we have $p_i = 0$. The coefficient of p_λ in $Z^c_{\widehat{\mathrm{Lie}}}$ is given by the residue d_λ:

$$d_\lambda = \int \mathrm{Lie}\circ Z^p_{\widehat{\mathrm{Lie}}}\prod_{i=1}^{r}\frac{dp_i}{p_i^{\lambda_i+1}}.$$

To compute d_λ, we use the same substitution $y_i = p_i\circ Z^p_{\widehat{\mathrm{Lie}}}$. It gives, expanding log:

$$d_\lambda = -\int\sum_{p\geq 1}\frac{\mu(p)}{p}\log(1-y_p)\prod_{i=1}^{r}\prod_{k,l\geq 1}(1-y_{kli})^{-\frac{i\lambda_i t^{ki}\mu(l)}{kli}}\left(1-\frac{t^i y_i}{1-y_i}\right)\frac{dy_i}{y_i^{\lambda_i+1}}$$
$$= \sum_{p\geq 1}\frac{\mu(p)}{p}\sum_{q\geq 1}\frac{1}{q}\int y_p^q\prod_{i=1}^{r}(1-y_i)^{-\varphi_i}\left(1-\frac{t^i y_i}{1-y_i}\right)\frac{dy_i}{y_i^{\lambda_i+1}}.$$

This gives the expected result.

□

3.2.3. *Link with* Λ. In this part, we link hypertrees decorated by $\widehat{\Sigma\,\mathrm{Lie}}$ and $\widehat{\mathrm{Lie}}$ with the operad Λ. We use the operad Pasc defined in the article of F. Chapoton [Cha02]. We recommend the reader to consult the reminder on cycle index series in the appendix 5 for the definition of the suspension Σ.

Proposition 3.17. *The hollow hypertrees decorated by* $\widehat{\Sigma\,\mathrm{Lie}}$ *and* $\widehat{\mathrm{Lie}}$ *are related by the following relation:*

$$Z^c_{\widehat{\mathrm{Lie}}}(t,p_1,p_2,\ldots) = \Sigma Z^c_{\widehat{\Sigma\,\mathrm{Lie}}}(-t,p_1,p_2,\ldots).$$

Proof. According to the corollary 1.38, the series $Z^c_{\widehat{\Sigma\,\mathrm{Lie}}}$ satisfies:

$$Z^c_{\widehat{\Sigma\,\mathrm{Lie}}}(t,p_1,p_2,\ldots) = \Sigma\,\mathrm{Lie}\circ\left(p_1(1+\mathrm{Comm})\circ(t\times Z^c_{\widehat{\Sigma\,\mathrm{Lie}}}(t,p_1,p_2,\ldots))\right).$$

Hence, applying the suspension and using the proposition 5.5 , we obtain:

$$\begin{aligned}\Sigma Z^c_{\widehat{\Sigma \mathrm{Lie}}}(t,p_1,p_2,\ldots) &= \mathrm{Lie}\circ\left(-p_1\times\Sigma(1+\mathrm{Comm})\circ(t\times\Sigma Z^c_{\widehat{\Sigma \mathrm{Lie}}}(t,p_1,p_2,\ldots))\right),\\ &= \mathrm{Lie}\circ\left(p_1(1+\mathrm{Comm})\circ(-t\times\Sigma Z^c_{\widehat{\Sigma \mathrm{Lie}}}(t,p_1,p_2,\ldots))\right).\end{aligned}$$

The last equation is the equation defining $Z^c_{\widehat{\mathrm{Lie}}}(-t,p_1,p_2,\ldots)$, so we get the expected result.

□

In an unpublished paper, V. Dotsenko proved that the operad Λ is Koszul. This result gives the following proposition.

Proposition 3.18. *Hollow hypertrees decorated by $\widehat{\Sigma \mathrm{Lie}}$ are related to Λ by:*

$$Z^c_{\widehat{\Sigma \mathrm{Lie}}}(t,p_1,p_2,\ldots,p_i,\ldots) = \frac{\Lambda(\frac{1}{t},tp_1,t^2p_2,\ldots,t^ip_i,\ldots)}{t}.$$

Proof. The corollary 1.38 give the following relation, as $\mathrm{Comm}\circ\Sigma\,\mathrm{Lie}=p_1$:

$$\begin{aligned}\left(Z^c_{\widehat{\Sigma \mathrm{Lie}}}\right)^{-1} &= \frac{\mathrm{Comm}}{1+\mathrm{Comm}\circ tp_1},\\ &= \left(\exp\left(\sum_{k\geq 1}\frac{p_k}{k}\right)-1\right)\times\exp\left(-\sum_{k\geq 1}\frac{t^kp_k}{k}\right).\end{aligned}$$

Now, as Λ is Koszul, it satisfies $\Sigma\,\mathrm{Pasc}\circ\Lambda=p_1$, which gives the relation:

$$\begin{aligned}tp_1 = {}&\exp\left(-\sum_{k\geq 1}\frac{p_k\circ\Lambda(t,p_1,\ldots,p_i,\ldots)}{k}\right)\\ &\times\left(\exp\left(\sum_{k\geq 1}\frac{t^kp_k\circ\Lambda(t,p_1,\ldots,p_i,\ldots)}{k}\right)-1\right).\end{aligned}$$

Hence, substituting t by $\frac{1}{t}$ in this expression, we get:

$$\begin{aligned}\frac{p_1}{t} = {}&\exp\left(-\sum_{k\geq 1}\frac{p_k\circ\Lambda(\frac{1}{t},p_1,\ldots,p_i,\ldots)}{k}\right)\\ &\times\left(\exp\left(\sum_{k\geq 1}\frac{p_k\circ\Lambda(\frac{1}{t},p_1,\ldots,p_i,\ldots)}{t^kk}\right)-1\right).\end{aligned}$$

Therefore, as $1=t\times\frac{1}{t}$, we obtain:

$$\begin{aligned}\frac{p_1}{t} &= \exp\left(-\sum_{k\geq 1}\frac{t^kp_k\circ\frac{1}{t}\Lambda(\frac{1}{t},p_1,\ldots,p_i,\ldots)}{k}\right)\\ &\quad\times\left(\exp\left(\sum_{k\geq 1}\frac{p_k\circ\frac{1}{t}\Lambda(\frac{1}{t},p_1,\ldots,p_i,\ldots)}{k}\right)-1\right)\\ &= \left(Z^c_{\widehat{\Sigma \mathrm{Lie}}}\right)^{-1}\circ\frac{1}{t}\Lambda(\frac{1}{t},p_1,\ldots,p_i,\ldots).\end{aligned}$$

Composing by tp_1, we obtain that $\frac{1}{t}\Lambda(\frac{1}{t},tp_1,\ldots,t^ip_i,\ldots)=\Sigma\Sigma_t\Lambda(\frac{1}{t},p_1,\ldots)$ is the inverse of $\left(Z^c_{\widehat{\Sigma \mathrm{Lie}}}\right)^{-1}$ for the plethystic substitution.

□

4. Generalizations: Bi-decorated hypertrees

4.1. **Definitions of bi-decorated hypertrees and rooted or edge-pointed variations of them.** In this part, we generalize the decoration of edges studied previously to the decoration of edges and neighbourhood of vertices.

Definition 4.1. Given two species or linear species $\mathcal{S}_e$ and $\mathcal{S}_v$, a *bi-decorated (rooted) hypertree* is obtained from a $\mathcal{S}_e$-edge-decorated (rooted) hypertree by choosing for every vertex v of H an element of $\mathcal{S}_v(E_v)$, where E_v is the set of edges containing v.

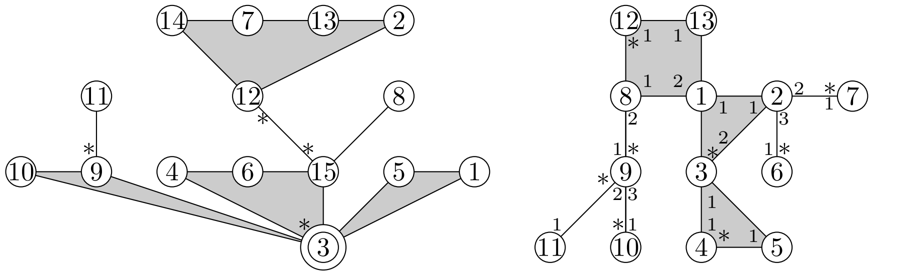
A *skew-bi-decorated rooted hypertree* is obtained from a $\mathcal{S}_e$-edge-decorated rooted hypertree H by choosing for every vertex v of H an element of $\mathcal{S}_v'(E_v)$, where E_v is the set of edges containing v.

Let us remark that when $\mathcal{S}_v$ is the species Comm, bi-decorated rooted hypertrees and skew-bi-decorated rooted hypertrees are the same type of hypertrees: edge-decorated rooted hypertrees.

The map which associates to a finite set I the set of bi-decorated hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-bi-decorated hypertree species and denoted by $\mathcal{H}_{\mathcal{S}_e,\mathcal{S}_v}$.The map which associates to a finite set I the set of bi-decorated rooted hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-bi-decorated rooted hypertree species and denoted by $\mathcal{H}^{Ap}_{\mathcal{S}_e,\mathcal{S}_v}$. The map which associates to a finite set I the set of skew-bi-decorated rooted hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-skew-bi-decorated rooted hypertree species and denoted by $\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}$.

Example 4.2. On the right side below is represented an example of bi-decorated hypertree with the neighbourhood of vertices decorated by the species of non-empty lists Assoc and edges decorated by the species of non-empty pointed sets Perm. The pointed vertex of each edge is represented with a star $*$ next to it. The order of edges around a vertex is given by the numbers around the vertex, next to the edges. For instance, around the vertex 9, the edge $\{8, 9\}$ is the first in the list, $\{10, 9\}$ is the second one and $\{11, 9\}$ is the third and last one.

On the left side, we have represented an example of bi-decorated rooted hypertree, with edges decorated by the species of cycle and the neighbourhood of vertices decorated by the species of non-empty pointed sets Perm. When the vertex is in only one edge, this edge is the pointed edge, otherwise, we put a star $*$ near the pointed edge of each vertex. For instance, in the neighbourhood of the vertex 15, the pointed edge is the edge $\{12, 15\}$, which is also pointed in the neighbourhood of the vertex 12.



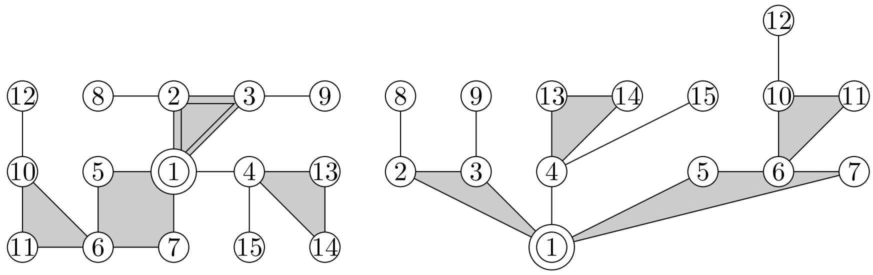
We now give definitions for edge-pointed bi-decorated hypertrees.

Definition 4.3. Given two species $\mathcal{S}_e$ and $\mathcal{S}_v$, a *bi-decorated edge-pointed (rooted) hypertree* (resp. *bi-decorated hollow hypertree*) is obtained from a $\mathcal{S}_e$-edge-decorated edge-pointed (rooted) hypertree (resp. a $\mathcal{S}_e$-edge-decorated hollow hypertree) H by choosing for every vertex v of H an element of $\mathcal{S}_v(E_v)$, where E_v is the set of edges containing v.

We can give an alternative definition. Given a vertex v, there is one edge containing v which is the nearest from the pointed or hollow edge of H: let us call it the *branch* of v. Then, a *bi-decorated edge-pointed (rooted) hypertree* (resp. *bi-decorated hollow hypertree*) is obtained from the hypertree H by choosing for every vertex v of H an element of $\mathcal{S}_v'(E_v)$, where E_v is the set of edges containing v different from the branch.

The map which associates to a finite set I the set of bi-decorated edge-pointed hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-edge-decorated edge-pointed hypertree species and denoted by $\mathcal{H}^a_{\mathcal{S}_e,\mathcal{S}_v}$. The map which associates to a finite set I the set of bi-decorated edge-pointed rooted hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-edge-decorated edge-pointed rooted hypertree species and denoted by $\mathcal{H}^{pa}_{\mathcal{S}_e,\mathcal{S}_v}$. The map which associates to a finite set I the set of bi-decorated hollow hypertrees on I is a species, called the $(\mathcal{S}_e, \mathcal{S}_v)$-edge-decorated hollow hypertree species and denoted by $\mathcal{H}^c_{\mathcal{S}_e,\mathcal{S}_v}$.

Example 4.4. We decorate the edge-pointed rooted hypertree on the left side of the example by the species of cycles on edges and around vertices. This give the planar hypertree on the right side of the example where the vertices and the edges are ordered in lists according to their place in the plane. The leftmost edge attached to the root is the pointed edge. For every vertex different from the root, the edge coming from the root to the vertex is the branch of the vertex and the other edges containing the vertex form a list.



4.2. **Relations.**

4.2.1. *Dissymetry principle.* We use the dissymetry principle of the proposition 1.35. As the decoration around vertices is defined similarly for all types of hypertrees, the dissymetry principle is still true on bi-decorated hypertrees.

Proposition 4.5 (Dissymmetry principle for bi-decorated hypertrees)**.** *Given two species $\mathcal{S}_e$ and $\mathcal{S}_v$, the following relation holds:*

$$\mathcal{H}_{\mathcal{S}_e,\mathcal{S}_v} + \mathcal{H}^{pa}_{\mathcal{S}_e,\mathcal{S}_v} = \mathcal{H}^{Ap}_{\mathcal{S}_e,\mathcal{S}_v} + \mathcal{H}^{a}_{\mathcal{S}_e,\mathcal{S}_v}. \tag{4.1}$$

4.2.2. *Functional equations.* The previous species are linked by the following proposition:

Proposition 4.6. *Let consider the species $\mathcal{S}_e$, such that $\mathcal{S}_e(\emptyset) = \emptyset$ and $\#\mathcal{S}_e(\{1\}) = 1$, and $\mathcal{S}_v$, such that $\mathcal{S}_v(\emptyset) = \emptyset$ and $\#\mathcal{S}_v(\{1\}) = 1$. The species $\mathcal{H}_{\mathcal{S}_e,\mathcal{S}_v}$, $\mathcal{H}^{Ap}_{\mathcal{S}_e,\mathcal{S}_v}$, $\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}$, $\mathcal{H}^{a}_{\mathcal{S}_e,\mathcal{S}_v}$, $\mathcal{H}^{pa}_{\mathcal{S}_e,\mathcal{S}_v}$ and $\mathcal{H}^{c}_{\mathcal{S}_e,\mathcal{S}_v}$ satisfy:*

$$t\mathcal{H}^{Ap}_{\mathcal{S}_e,\mathcal{S}_v} = X + X \times \left(\mathcal{S}_v \circ t\mathcal{H}^{c}_{\mathcal{S}_e,\mathcal{S}_v}\right), \tag{4.2}$$

$$t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v} = X + X \times \left((\mathcal{S}_v' - 1) \circ t\mathcal{H}^{c}_{\mathcal{S}_e,\mathcal{S}_v}\right), \tag{4.3}$$

$$\mathcal{H}^{c}_{\mathcal{S}_e,\mathcal{S}_v} = (\mathcal{S}_e' - 1) \circ t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}, \tag{4.4}$$

$$\mathcal{H}^{a}_{\mathcal{S}_e,\mathcal{S}_v} = (\mathcal{S}_e - X) \circ t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}, \tag{4.5}$$

$$\mathcal{H}^{pa}_{\mathcal{S}_e,\mathcal{S}_v} = \mathcal{H}^{c}_{\mathcal{S}_e,\mathcal{S}_v} \times t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}. \tag{4.6}$$

Proof.
- The case of only one vertex has already been established in proposition 1.37. Otherwise, we separate the label of the root: it gives X. It remains an hypertree with a gap contained in different edges, at least one. By definition, those edges has an $\mathcal{S}_v$-structure. Forgetting this structure,we obtain a non-empty set of hollow hypertree with edges decorated by $\mathcal{S}_e$ and vertices decorated by $\mathcal{S}_v$.
- This case only differs from the preceding case by the structure around the root, which is a $\mathcal{S}_v'$-structure and not a $\mathcal{S}_v$-structure.
- The third relation is obtained by pointing the vertices in the hollow edge and breaking it: we obtain a non-empty forest of rooted edge-decorated hypertrees. The set of roots is a $\mathcal{S}_e'$-structure and induces this structure on the set of trees: we obtain a $\mathcal{S}_e'$-structure in which all elements are rooted edge-decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is an isomorphism of species.
- The fourth relation is obtained by pointing the vertices in the pointed edge and breaking it: we obtain a non-empty forest of rooted edge-decorated hypertrees. The set of roots is a $\mathcal{S}_e$-structure and induces this structure on the set of trees: we obtain a $\mathcal{S}_e$-structure in which all elements are rooted edge-decorated hypertrees. As this operation is reversible and does not depend on the labels of the hollow hypertree, this is an isomorphism of species.
- The last equation is obtained by decorating around the vertices of the two parts of the last equality of proposition 1.37.

□

Corollary 4.7. *Using the equations* (4.3) *and* (4.4) *of the proposition, we obtain:*

$$\mathcal{H}^c_{\mathcal{S}_e,\mathcal{S}_v} = (\mathcal{S}_e' - 1) \circ \left(X + X \times \left((\mathcal{S}_v' - 1) \circ t\mathcal{H}^c_{\mathcal{S}_e,\mathcal{S}_v}\right)\right), \tag{4.7}$$

$$t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v} = X + X \times \left((\mathcal{S}_v' - 1) \circ (t(\mathcal{S}_e' - 1)) \circ t\mathcal{H}^{Bp}_{\mathcal{S}_e,\mathcal{S}_v}\right). \tag{4.8}$$

4.3. **Friendly cases of Bi-decorated hypertrees and link with the hypertree poset.** In the article [Oge12], we link the character of the action of the symmetric group on Whitney homology of the hypertree poset with the symmetric function HAL defined by F. Chapoton in his article [Cha07]. We now give a combinatorial interpretation of HAL in terms of bi-decorated hypertrees.

As this series has been inspired by the one for *cyclic hypertrees*, we first study the link between bi-decorated hypertrees and this series.

4.3.1. *Cyclic hypertrees. Cyclic hypertrees* are defined as hypertrees with, for every vertex v, a cyclic order on the edges containing v. The associated species is denoted by HAC. This definition corresponds to the one of bi-decorated hypertrees obtained by taking the species Comm for $\mathcal{S}_e$ and the cycle species Cycle for $\mathcal{S}_v$.

As for usual hypertrees, we consider rooted cyclic hypertrees, edge-pointed cyclic hypertrees and edge-pointed rooted cyclic hypertrees and respectively write HAC^p, HAC^a and HAC^{pa} for the associated species. Considering the definitions, HAC^a and $\mathcal{H}^a_{\mathrm{Comm,Cycle}}$, and also HAC^{pa} and $\mathcal{H}^{pa}_{\mathrm{Comm,Cycle}}$ are isomorphic because in these definitions, pointing and decoration commute. Let us compare the relations to find once more a link between the different types of hypertrees.

On the one hand, the species defined by F. Chapoton satisfy the following relations:

$$\mathrm{HAC}^{pa} = \frac{X}{t} \times (\mathrm{Assoc} \circ t\,\mathrm{YC}),$$

with YC defined by:

$$\begin{aligned}
\mathrm{YC} &= \mathrm{Comm} \circ (X + t\,\mathrm{HAC}^{pa}),\\
\mathrm{HAC}^{a} &= (\mathrm{Comm} - X) \circ (X + t\,\mathrm{HAC}^{pa}),\\
\mathrm{HAC}^{p} &= \frac{X}{t}\,\mathrm{Cycle} \circ t\,\mathrm{YC},\\
\mathrm{HAC} &= \mathrm{HAC}^{a} + \mathrm{HAC}^{p} - \mathrm{HAC}^{pa}.
\end{aligned}$$

Whereas, on the other hand, the bi-decorated hypertrees satisfy:

$$\begin{aligned}
t\mathcal{H}^{Ap}_{\mathrm{Comm},\mathrm{Cycle}} &= X + X \times \left(\mathrm{Cycle} \circ t\mathcal{H}^{c}_{\mathrm{Comm},\mathrm{Cycle}}\right),\\
t\mathcal{H}^{Bp}_{\mathrm{Comm},\mathrm{Cycle}} &= X + X \times \left(\mathrm{Assoc} \circ t\mathcal{H}^{c}_{\mathrm{Comm},\mathrm{Cycle}}\right),\\
\mathcal{H}^{c}_{\mathrm{Comm},\mathrm{Cycle}} &= \mathrm{Comm} \circ t\mathcal{H}^{Bp}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathcal{H}^{a}_{\mathrm{Comm},\mathrm{Cycle}} &= (\mathrm{Comm} - X) \circ t\mathcal{H}^{Bp}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathcal{H}^{pa}_{\mathrm{Comm},\mathrm{Cycle}} &= \mathcal{H}^{c}_{\mathrm{Comm},\mathrm{Cycle}} \times t\mathcal{H}^{Bp}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathcal{H}_{\mathrm{Comm},\mathrm{Cycle}} &= \mathcal{H}^{Ap}_{\mathrm{Comm},\mathrm{Cycle}} + \mathcal{H}^{a}_{\mathrm{Comm},\mathrm{Cycle}} - \mathcal{H}^{pa}_{\mathrm{Comm},\mathrm{Cycle}}.
\end{aligned}$$

Comparing these equations, we obtain the following relations:

$$\begin{aligned}
\mathrm{HAC}^{a} &= \mathcal{H}^{a}_{\mathrm{Comm},\mathrm{Cycle}},\\
X + t\,\mathrm{HAC}^{pa} &= t\mathcal{H}^{Bp}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathrm{YC} &= \mathcal{H}^{c}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathrm{HAC}^{pa} &= \mathcal{H}^{pa}_{\mathrm{Comm},\mathrm{Cycle}},\\
\mathrm{HAC}^{p} &= \frac{X}{t} + \mathcal{H}^{Ap}_{\mathrm{Comm},\mathrm{Cycle}}.
\end{aligned}$$

Let us now study the case of the HAL series.

4.3.2. *The hypertree poset.* The series HAL, HAL^{p}, HAL^{a} and HAL^{pa} defined in [Cha07] satisfy the following relations:

$$\begin{aligned}
\mathrm{HAL}^{pa} &= \frac{p_1}{t} \times \left(\Sigma\,\mathrm{Assoc} \circ t\,\mathrm{Comm} \circ (p_1 + (-t)\,\mathrm{HAL}^{pa})\right),\\
\mathrm{HAL}^{p} &= \frac{p_1}{t}\left(\Sigma\,\mathrm{Lie} \circ t\,\mathrm{Comm} \circ (p_1 + (-t)\,\mathrm{HAL}^{pa})\right),\\
\mathrm{HAL}^{a} &= (\mathrm{Comm} - p_1) \circ (p_1 + t\,\mathrm{HAL}^{pa}),\\
\mathrm{HAL} &= \mathrm{HAL}^{a} + \mathrm{HAL}^{p} - \mathrm{HAL}^{pa}.
\end{aligned}$$

We link these series with the cycle index series of bi-decorated hypertrees obtained by taking the species Comm for $\mathcal{S}_e$ and the species $\Sigma\,\mathrm{Lie}$ for $\mathcal{S}_v$

Whereas the bi-decorated hypertrees satisfy:

$$\begin{aligned}
t\mathcal{H}^{p}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} &= X + X \times \left(\Sigma\,\mathrm{Lie} \circ t\,\mathrm{Comm} \circ t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}\right),\\
t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} &= X + X \times \left(\Sigma\,\mathrm{Assoc} \circ t\,\mathrm{Comm} \circ t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}\right),\\
\mathcal{H}^{a}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} &= (\mathrm{Comm} - X) \circ t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}},\\
\mathcal{H}^{pa}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} &= \left(\mathrm{Comm} \circ t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}\right) \times t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}},\\
\mathcal{H}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} &= \mathcal{H}^{p}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} + \mathcal{H}^{a}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}} - \mathcal{H}^{pa}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}.
\end{aligned}$$

Comparing these equations, we obtain the following relations:

$$\begin{aligned} p_1 + (-t)\,\mathrm{HAL}^{pa} &= t\mathcal{H}^{sp}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}, \\ \mathrm{HAL}^{a} &= \mathcal{H}^{a}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}, \\ \mathrm{HAL}^{pa} &= \mathcal{H}^{pa}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}, \\ \mathrm{HAL}^{p} &= \frac{p_1}{t} + \mathcal{H}^{p}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}, \\ \mathrm{HAL} &= \frac{p_1}{t} + \mathcal{H}_{\mathrm{Comm},\Sigma\,\mathrm{Lie}}. \end{aligned}$$

Therefore, the character of the action of the symmetric group on the Whitney homology of the hypertree poset is the same as the character of the action of the symmetric group on the set of hypertrees whose vertices have their neighbourhood decorated by $\Sigma\,\mathrm{Lie}$.

5. Reminder on cycle index

Let F be a species. We can associate a formal power series to it: its cycle index. The reader can consult [BLL98] for a reference on this subject. This formal power series is a symmetric function defined as follows:

Definition 5.1. The **cycle index** of a species F is the formal power series in an infinite number of variables $(p_1, p_2, p_3, \ldots)$ defined by:

$$\mathbf{C}_F(p_1, p_2, p_3, \ldots) = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\sigma\in\mathfrak{S}_n} \# F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

where F^σ stands for the set of F-structures fixed under the action of σ and where σ_i is the number of cycles of length i in the decomposition of σ into disjoint cycles.

For instance, the cycle index series of the species X of singletons is p_1.

We can define the following operations on cycle index series.

Definition 5.2. The operations $+$ and $\times$ on cycle index series are the same as on formal series.

For $f = f(p_1, p_2, \ldots)$ and $g = g(p_1, p_2, \ldots)$, plethystic substitution $f \circ g$ is defined by:

$$f \circ g(p_1, p_2, \ldots) = f(g(p_1, p_2, p_3, \ldots), g(p_2, p_4, p_6, \ldots), \ldots, g(p_k, p_{2k}, p_{3k}, \ldots), \ldots).$$

It is left-linear.

These operations satisfy:

Proposition 5.3. *Let F and G be two species. Their cycle index series satisfy:*

$$\begin{aligned} \boldsymbol{C}_{F+G} &= \boldsymbol{C}_F + \boldsymbol{C}_G, & \boldsymbol{C}_{F\times G} &= \boldsymbol{C}_F \times \boldsymbol{C}_G, \\ \boldsymbol{C}_{F\circ G} &= \boldsymbol{C}_F \circ \boldsymbol{C}_G, & \boldsymbol{C}_{F'} &= \frac{\partial \boldsymbol{C}_F}{\partial p_1}. \end{aligned}$$

Moreover, we define the following operation:

Definition 5.4. The *suspension* Σ_t of a cycle index $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma_t f = -\frac{1}{t} f(-tp_1, -t^2 p_2, -t^3 p_3, \ldots).$$

By convention, we will write Σ for the suspension in $t = 1$.

The suspension satisfies:

Proposition 5.5. *Let f and g be to cycle index series. They satisfy:*

- $\Sigma\,(f \circ g) = \Sigma f \circ \Sigma g$,
- $\Sigma\,(f \times g) = -\Sigma f \times \Sigma g$,
- $-\Sigma f = f \circ (-p_1)$.

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